Equatorial Waves

Introduction

Each of the equatorial waves represents a solution to the shallow water equation set. To best understand these waves, it is fruitful to step through the derivation of the shallow water equation set. Herein, we first introduce the relevant equations and the relevant physical assumptions that we make in order to simply this equation set. Subsequently, we linearize these equations about a base and perturbation state. We then assume a wave-like solution to the remaining free variables of the linearized system and obtain the general dispersion relation for the shallow water equation system. Solutions to the shallow water system for each wave type are obtained and described. We close our discussion of equatorial waves with a brief discussion of how these waves are monitored and how they are manifest in larger-scale modes of tropical variability.

Key Questions

- What are Kelvin waves and why are they important?
- What is an equatorial Rossby wave and why is it important?
- What are inertia-gravity and mixed Rossby-gravity waves and why are they important?
- How do these waves modulate convective activity and large-scale tropical phenomena?

An Introduction to Equatorial Waves

There are four primary types of equatorial waves that we are concerned with in this course. These are the *Kelvin*, *equatorial Rossby*, *mixed Rossby-gravity*, and, to lesser extent, *inertia-gravity* waves. Each of these waves represent specific solutions to the shallow water equation system, the assumptions inherent to which are described as the solution for each wave type is detailed. We first describe each wave type, focusing almost exclusively upon the atmospheric manifestations of these waves. We then step through the derivation of each of these wave types, focusing upon their common mathematical basis and the inherent physical differences between them. We close by discussing how these waves modulate and are modulated by deep, moist convection and how we can monitor and detect such waves.

An Introduction to the Predominant Equatorial Wave Modes

Kelvin Waves

Kelvin waves are large-scale waves that propagate along a physical boundary such as a mountain range or coastline. In the tropics, the northern and southern hemispheres each act as a trapping barrier, such that equatorial Kelvin waves are said to be “equatorially-trapped” waves. Kelvin waves are thought to be important to the El Nino Southern Oscillation (ENSO), Madden-Julian Oscillation (MJO), and Quasi-Biennial Oscillation (QBO). They exert a significant influence on deep, moist convection within 10° latitude of the equator with the greatest such impact typically observed in the eastern Indian and central Pacific Ocean. Their impacts on deep, moist convection over Africa, South America, and the western Indian Ocean exhibit strong seasonal variability. The westerly wind bursts that often accompany Kelvin waves are occasionally caused by tropical cyclones.

Kelvin waves have a length scale of approximately 2000 km. They have a direct impact on deep, moist convection within approximately 10° latitude of the Equator. As will be demonstrated later, there is no meridional component of velocity that is associated with a Kelvin wave. At any given location, Kelvin waves have a return period of about 6-7 days. Convectively-coupled Kelvin waves, or those associated with and linked to deep, moist convection, propagate eastward with a phase speed between 12-25 m s^{-1}. Dry Kelvin waves propagate at a faster velocity whereas convectively-coupled Kelvin waves propagate at a slower velocity. Note that in general, for all types of equatorial waves, deep, moist convection results in a slowing of the propagation speed of the wave.

Equatorial Rossby Waves

In the mid-latitudes, Rossby waves arise out of meridional variability in the potential vorticity field. In the tropics, this can be generalized to simply the meridional variability in the Coriolis parameter. Equatorial Rossby waves are associated with twin vortices on either side of the equator; such vortices occur most often in the Indian Ocean and western Pacific Ocean. The direct impacts of equatorial Rossby waves are strongest over the Asian monsoon and West Pacific warm pool. These vortices have a length scale of approximately 1000 km and can repeat over very long zonal distances of up to 10,000 km. Equatorial Rossby waves last on the order of days to weeks. Convectively-coupled (dry) equatorial Rossby waves move westward with a phase speed between 5-7 m s^{-1} (10-20 m s^{-1}).

Mixed Rossby-Gravity Waves

These waves are forced by and, subsequently, force deep, moist convection through their ties to buoyancy. Buoyancy and the meridional variation in the Coriolis parameter, the two restoring mechanisms for mixed Rossby-gravity waves, helps to give them characteristics of both inertia-gravity waves (which are tied to buoyancy) and equatorial Rossby waves (which are tied to meridional variation in the Coriolis parameter). These waves are generally tilted northwest-southeast across the equator. Mixed Rossby-gravity waves occur most frequently across the equatorial western and central Pacific and during summer and autumn in the Northern Hemisphere. The horizontal length scale of mixed Rossby-gravity waves is approximately 1000 km, or of similar magnitude to equatorial Rossby waves. They have a period of 4-5 days and move westward at a phase speed of approximately 8-10 m s^{-1}. It is the differing horizontal structures, length scales, and propagation speeds that enable each type of wave to be detected from satellite or conventional observations.

In the tropics, each wave type is forced by heating, whether associated with latent heat release due to deep, moist convection or otherwise. The vertical motions that result from such heating, such as we have considered in previous lectures, help to locally counterbalance the heating and maintain the weak horizontal temperature gradients characteristic of the tropics. In the following, for simplicity, we will derive solutions to the predominant equatorial waves in an unforced (i.e., no heating) framework. This enables us to consider their basic structures. We will later consider how deep, moist convection and associated latent heat release modify this structure (e.g., the slowing of wave propagation described earlier). To start the derivation process, we introduce the shallow water system of equations.

Derivation of the Equatorial Wave Modes: The Shallow Water Equations

For the shallow water system, there are three relevant equations. The first, the equation of motion, describes the two-dimensional *(x,y)* motion of the system. The second, the hydrostatic equation, describes the nature of vertical motions within the system. Together, these equations describe the conservation of momentum (not to be confused with our earlier discussion on the conservation of absolute angular momentum). The third, the continuity equation, describes the conservation of mass within the system. These equations take the form:

(1) $$\frac{D\mathbf{v}}{Dt} + f\hat{k} \times \mathbf{v} = -\frac{1}{\rho}\nabla p$$

(2) $$\frac{\partial p}{\partial z} = -\rho g$$

(3) $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{\partial w}{\partial z}$$

where *D*/*Dt* is the total derivative, boldface variables denote vectors, and all variables adhere to typical meteorological conventions.

There are a number of approximations inherent to the shallow water equation system. First and foremost, the concept of "shallow water" means that we assume that the vertical scale is much smaller than the horizontal scale. For equatorial waves, this is a fair assumption: vertical scales are on the order of 15-40 km whereas horizontal scales are on the order of 1000 km or more. We then assume that the atmosphere can be approximated by two distinct layers, each with constant density (ρ_1 in the lower layer, ρ_2 in the upper layer). Given that we have invoked the hydrostatic approximation in our system of equations above, this enables us to say that the horizontal pressure gradient in each layer is independent of height. This can be demonstrated by taking the horizontal derivative of the hydrostatic approximation and commuting the derivatives. Waves in the shallow water system are said to be of finite amplitude (i.e., quasi-linear). We assume that the equations are incompressible, wherein the density is conserved following the motion such that the total derivative of the density is equal to zero; this assumption means that sound waves cannot be possible solutions. We neglect friction. We assume that the environment is stably stratified, such that the density in the lower layer is greater than that in the upper layer (i.e., $\rho_1 > \rho_2$, such that $p_1 > p_2$). Finally, we assume that there is no horizontal pressure gradient in the *upper* layer.

As we proceed, please refer to the slide in the "Equatorial Waves" lecture materials depicting the basic structure of a two-dimensional shallow water system. In order to simplify our set of equations above, particularly (1), we desire an expression for the horizontal pressure gradient in the *lower* layer. In other words, we want to know how pressure varies between points B and A within the model depicted within the lecture materials. With no horizontal pressure gradient in the upper layer, the pressure along the interface between the two layers (above point B) is equivalent to that within the upper layer (above point A). We first assume that differences in pressure along and near the interface between the upper and layer are small (i.e., finite and *infinitesimal*). The pressure at point A is a function of a displacement in pressure associated with a downward forced upper layer and the pressure at point B is a function of a displacement in pressure associated with an upward forced lower layer. We will refer to these displacements as δp_2 and δp_1, respectively. Thus, the pressure at points A and B can be expressed as:

(4a) $$A: p + \delta p_2$$

(4b) $$B: p + \delta p_1$$

We can use the hydrostatic relationship to re-write (4) in terms of the density within each layer and the displacement in height associated with the wave, such that:

(5a) $$p + \delta p_2 = p + \rho_2 g \delta z$$

(5b) $$p + \delta p_1 = p + \rho_1 g \delta z$$

It should be noted in (5) above that the leading negative associated with the hydrostatic approximation is folded into the vertical displacement variable.

Next, let the height of the interface at the point above point B be equal to h_2. Similarly, let the height of the interface at point A be equal to h_1. In this case, δz is merely equal to $h_2 - h_1$. However, let us consider the case where the distance between points B and A, ∂x, is infinitesimally small (i.e., δx -> 0). In this case,

(6) $$\delta z = \frac{h_2 - h_1}{\partial x}\delta x = \frac{\partial h}{\partial x}\delta x$$

If we substitute (6) into (5), divide all by δx, subtract the expression for point A (e.g., from 5a) from that for point B (e.g., from 5b), we obtain the following:

(7) $$\lim_{\delta x \to 0}\left[\frac{(p + \delta p_1) - (p + \delta p_2)}{\delta x}\right] = g\delta\rho\frac{\partial h}{\partial x}$$

where $\delta\rho$ is equal to ρ_2-ρ_1. The corresponding expression for the meridional pressure (and height) gradients can be obtained in a similar manner and takes on an identical form (except in terms of y). These expressions are akin to saying that horizontal gradients in pressure are equivalent to horizontal gradients in the depth (h) of the fluid.

If we substitute the right-hand side of (7) and accompanying expression for $\partial h/\partial y$ into (1) above and expand into the full u-momentum and v-momentum equations, we obtain:

(8a) $$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = fv - g\frac{\delta\rho}{\rho_1}\frac{\partial h}{\partial x}$$

(8b) $$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -fu - g\frac{\delta\rho}{\rho_1}\frac{\partial h}{\partial y}$$

We wish to make one further simplification to (8) before manipulating the continuity equation given by (3). Let us express the fluid depth h, which is a function of (x,y,t), in terms of a constant height plus a perturbation height, i.e.,

(9) $$h(x,y,t) = H + h'(x,y,t)$$

In the above, H is defined as the equivalent depth and is proportional to the stability. It impacts the vertical wavenumber and thus the vertical structure and depth of the wave. We find that H can take values between 10-500 m such that the resultant vertical wavelength is large (5-50 km). More commonly, H is said to be between 200-400 m, resulting in vertical wavelengths of 25-40 km. Thus, the fluid depth for our equatorial waves can be quite large with vertical wave structure found throughout the depth of that fluid! This becomes important when we describe the dynamics of the Quasi-Biennial Oscillation in subsequent lectures.

If we substitute (9) into (8), the constant term drops out such that:

(10a) $$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = fv - g\frac{\delta\rho}{\rho_1}\frac{\partial h'}{\partial x}$$

(10b) $$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -fu - g\frac{\delta\rho}{\rho_1}\frac{\partial h'}{\partial y}$$

Next, we wish to manipulate the continuity equation given by (3) above. We wish to integrate both sides of (3) from the ground, where the vertical motion must be equal to zero, to h, the fluid height. Since the pressure gradient expressed in (7) is independent of height (i.e., not a function of z), (10) makes it clear that both u and v are independent of height (and thus are not functions of z) presuming that they were not functions of height at the initial time t=0. As a result, our integration is simplified. First, the left-hand side of (3):

(11a) $$\int_{z'=0}^{z'=h}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)dz' = h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

Next, the right-hand side of (3):

(11b) $$-\int_{z'=0}^{z'=h}\left(\frac{\partial w}{\partial z}\right)dz' = -\left[w(z=h) - w(z=0)\right]$$

Since our lower boundary is flat and rigid, $w(z=0)$ is equal to 0. Meanwhile, w at the fluid interface is merely a reflection of the vertical movement of the fluid interface itself and can be expressed as:

(12) $$w(h) = \frac{Dh}{Dt} = \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y}$$

If we simplify (11b), substitute (12) in to (11b), and set the resulting expression equal to (11a), we obtain:

(13) $$-h\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=\frac{\partial h}{\partial t}+u\frac{\partial h}{\partial x}+v\frac{\partial h}{\partial y}$$

In the process of obtaining (13), we multiplied both sides by -1 to bring the leading negative on (11b) over to the left-hand side of the continuity equation.

Finally, substitute (9) into (13) to obtain:

(14) $$-(H+h')\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=\frac{\partial h'}{\partial t}+u\frac{\partial h'}{\partial x}+v\frac{\partial h'}{\partial y}$$

This equation states that the movement of the fluid interface (right-hand side of (14)) is equal to the depth of the lower fluid times the convergence in the lower fluid. Either greater or deeper convergence will lead to greater movement of the fluid interface. Thus, equations (10) and (14) describe our shallow water set of equations.

Next, we wish to simplify the Coriolis term in (10). We introduce the concept of a Beta plane, where the range of latitudes under consideration is said to be sufficiently small as to enable the meridional variation in the Coriolis parameter to be treated as a linear, rather than non-linear (e.g., sin Φ), function of y. This simplifies the solving of the shallow water equation set. This approximation results from performing a Taylor series expansion on f and keeping only the first two terms, such that:

(15) $$f=f_0+\beta y, \beta=\frac{\partial f}{\partial y}$$

Recalling that $f = 2\Omega\sin\Phi$, where Φ is latitude, β is thus given by $(2\Omega\cos\Phi)/a$, where a is the radius of the Earth and generally taken constant at 6.37×10^6 m. If we restrict (15) to the equator, such that f_0 is zero, then $f = \beta y$. If we employ the small angle approximation (i.e., Φ small), then $\cos\Phi \approx 1$ and $\beta = 2\Omega/a = 2.3 \times 10^{-11}$ m^{-1} s^{-1}. Rather than input this numerical value into (10) and (14), however, we substitute $f = \beta y$ into these equations to obtain:

(16a) $$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\beta y v-g\frac{\delta\rho}{\rho_1}\frac{\partial h'}{\partial x}$$

(16b) $$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\beta y u-g\frac{\delta\rho}{\rho_1}\frac{\partial h'}{\partial y}$$

(17) $$-(H+h')\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=\frac{\partial h'}{\partial t}+u\frac{\partial h'}{\partial x}+v\frac{\partial h'}{\partial y}$$

where (17) is identical to (14) as there are no Coriolis terms in (14).

Finally, we linearize (16) and (17) about a background state that is initially at rest. In this process, we assume that the three variables represented within the system given by (16) and (17) – u, v, and h –

can each be partitioned into mean (or background flow) and perturbation (or wave flow) components. These take the form:

(18a) $$u(x,y,t) = \overline{u}(x,y,t) + u'(x,y,t)$$

(18b) $$v(x,y,t) = \overline{v}(x,y,t) + v'(x,y,t)$$

(18c) $$h(x,y,t) = H + \overline{h}(x,y,t) + h'(x,y,t)$$

Equation (18c) has three terms as our initial definition of *h* (given by (9)) did not allow for spatial or temporal variance in the base-state *H*. Note that in linearization theory, the perturbation fields *u'*, *v'*, and *h'* must be small. The mean terms in (18) can all be taken as equal to zero if we assume a background state with no horizontal or vertical motion.

It is worth noting at this point that equatorial waves in the shallow water system are expressed in terms of three dimensions: *x*, *y*, and *t*. However, it is possible to formulate a vertical structure equation that describes the vertical structure of these waves. This equation is a second order partial differential equation and is a function of the vertical wavenumber *m*, itself largely dependent upon stability (e.g., as enters through the equivalent depth *H*). As noted above, the vertical wavelength of these equatorial waves is typically on the order of 25-40 km. The vertical structure equation can be used to provide detail of a wave's structure over that wavelength. For our purposes, however, it is most important to know that equatorial waves do contain vertical structure.

If we substitute (18) into (16) and (17) and simplify, we obtain:

(19a) $$\frac{\partial u'}{\partial t} - \beta y v' = -g'\frac{\partial h'}{\partial x}$$

(19b) $$\frac{\partial v'}{\partial t} + \beta y u' = -g'\frac{\partial h'}{\partial y}$$

(20) $$\frac{\partial h'}{\partial t} + H\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right) = 0$$

where we have moved the β terms to the left-hand side of the equations. We have also substituted an "effective gravity" term *g'* into the right-hand side of (16), as defined by:

(21) $$g' = g\frac{\delta\rho}{\rho_1}$$

Note that the terms involving products of linearized variables in (16) and (17) vanish. Substitution of (18) into these terms results in four terms. Three of these terms involve the mean fields in some way. As they are zero by definition, such terms vanish. The remaining term involves the product of two perturbation fields. As perturbations are necessarily small, the product of perturbations is sufficiently small as to be negligible. Thus, each of these four terms vanish.

Equations (19) and (20) can be used to formulate a perturbation potential vorticity equation for the shallow water system. Specifics of this formulation are provided within Appendix A1.5 of Chapter 4 of *An Introduction to Tropical Meteorology, 2nd Edition*. The basic procedure involves formulating a vorticity equation from (19) and manipulating it using (20) and inherent properties of the system to re-write that equation in terms of perturbation potential vorticity q. This perturbation potential vorticity has the form:

(22) $$q = \frac{\zeta + f}{h}$$

and is said to be a conserved quantity following the motion.

We are now ready to explore solutions to the shallow water equation set given by (19) and (20). This set has three free variables: u', v', and h'. We assume wave-like solutions for each of these three variables as follows:

(23a) $$u'(x,y,t) = U(y) * \exp(i(kx - \omega t))$$

(23b) $$v'(x,y,t) = V(y) * \exp(i(kx - \omega t))$$

(23c) $$h'(x,y,t) = H_w(y) * \exp(i(kx - \omega t))$$

The functions given by U, V, and H_w are the amplitudes of the wave function for each variable. These amplitudes vary only in the north-south direction (i.e., as functions of y). k is the zonal wavenumber, ω is the frequency of the wave (equal to the number of times the wave passes a given point per second, and thus related to its propagation), and i is equal to the square root of -1. Before substituting (23) into (19) and (20), it is helpful to note that derivatives of (23) with respect to x and t have special forms given by:

(24a) $$\frac{\partial}{\partial x} = ik$$

(24b) $$\frac{\partial}{\partial t} = -i\omega$$

These arise because $U(y)$, $V(y)$, and $H_w(y)$ are all not functions of x or t and because the derivative of an exponential function is equal to the derivative of the exponential multiplied by the exponential function. Because each term of (19) and (20) will contain this exponential function when substituting in (23), we drop it in (24) for simplicity, noting that we will divide through by the exponential function after taking its derivative when substituting (23) into (19) and (20).

Performing this substitution, we obtain:

(25a) $$-i\omega U - \beta y V = -ikg' H_W$$

(25b) $$-i\omega V + \beta y U = -g'\frac{\partial H_W}{\partial y}$$

(26) $$-i\omega H_w + H\left(ikU + \frac{\partial V}{\partial y}\right) = 0$$

The set of equations given by (25) and (26) has three unique variables, *U*, *V*, and H_w, each of which are functions only of *y*. We now wish to simplify this system into one of two equations for two variables, after which we will simplify further into a system of one equation for one variable. First, we solve (25a) for *U*:

(27) $$U = -\frac{\beta y V}{i\omega} + \frac{kg' H_W}{\omega}$$

We then substitute (27) into (25b), multiply all terms by -*iω*, group like terms (specifically, the *V* terms), and rearrange slightly to obtain:

(28) $$(B^2 y^2 - \omega^2)V - ikg'\beta y H_W - i\omega g'\frac{\partial H_W}{\partial y} = 0$$

Similarly, we then substitute (27) into (26), multiply all terms by *iω*, group the like *V* and H_w terms, and rearrange slightly to obtain:

(29) $$(\omega^2 - Hk^2 g')H_W + iH\omega(\frac{\partial V}{\partial y} - \frac{k\beta y}{\omega}V) = 0$$

We follow a similar procedure to reduce (28) and (29) into a single equation, whereby we solve (29) for H_w and substitute the result into (28) to obtain a single equation for *V*. H_w and its derivative with respect to *y* are given by the following:

(30a) $$H_W = \frac{-i\omega H\frac{\partial V}{\partial y} + ikH\beta y V}{\omega^2 - g' Hk^2}$$

(30b) $$\frac{\partial H_W}{\partial y} = \frac{-i\omega H\frac{\partial^2 V}{\partial y^2} + ikH\beta y\frac{\partial V}{\partial y} + ikH\beta V}{\omega^2 - g' Hk^2}$$

where we have made use of the product rule for derivatives in obtaining (30b) from (30a). Substituting (30) into (28) enables us to obtain a second order partial differential equation for *V*. This substitution leaves us with three sets of terms: *V*, its first derivative with respect to *y*, and its second derivative with respect to *y*. Specifically, these terms take the form:

(31a) $$V\left[\left(\beta^2 y^2-\omega^2\right)+\frac{k\omega g' h\beta}{\omega^2-Hk^2 g'}+\frac{k^2 g'\beta^2 y^2 H}{\omega^2-Hk^2 g'}\right]$$

(31b) $$\frac{\partial V}{\partial y}\left[\frac{-kg' H\omega\beta y}{\omega^2-Hk^2 g'}+\frac{kg' H\omega\beta y}{\omega^2-Hk^2 g'}\right]=0$$

(31c) $$\frac{\partial^2 V}{\partial y^2}\left[\frac{-\omega^2 g' H}{\omega^2-Hk^2 g'}\right]$$

If you multiply (31a) by $\omega^2-Hk^2 g'$, cancel opposing terms, subsequently divide by $-\omega^2 g' H$, perform minor rearrangement, and combine with (31c), the aforementioned second-order PDE for *V* is obtained. This PDE is given by:

(32) $$\frac{\partial^2 V}{\partial y^2}+\left(\frac{\omega^2}{g' H}-\frac{\beta^2 y^2}{g' H}-k^2-\frac{\beta k}{\omega}\right)V=0$$

Assuming that solution(s) for *V* are known, they can be used with (30a) to obtain solution(s) for H_w and, subsequently, with (27) for *U*.

Now, we are ready to consider the full complexity given by (32). Herein, we require solutions where *V* approaches zero as *y*, the distance from the equator, grows increasingly large (i.e., approaches ±∞). This statement aids in constraining our solutions to have maximum amplitude near the equator and to decay north and south from there. In his seminal work on equatorial waves, Matsuno (1966) demonstrated that the solutions for (32) only satisfy this condition if there are a finite odd integer number of waves present in the meridional direction, i.e.,

(33a) $$\frac{\sqrt{g' H}}{\beta}\left(\frac{\omega^2}{g' H}-k^2-\frac{\beta k}{\omega}\right)=2n+1$$

Or, put in a way that is perhaps more apparent from a consideration of the coefficient on (32),

(33b) $$\frac{\omega^2}{g' H}-\frac{\beta}{\sqrt{g' H}}-k^2-\frac{\beta k}{\omega}=\frac{2n\beta}{\sqrt{g' H}}$$

Equation (33a) is obtained from a consideration of the coefficient on *V* in (32) in the context of *y* approaching ±∞ and how it modulates the solution to the second-order PDE in *V* given by (32). The value *n* is a general wavenumber and is equal to 0, 1, 2, 3, and so on.

Equation (33a) relates the frequency ω and zonal wavenumber *k* for all possible wave solutions (n = 0, 1, 2, 3…) to the shallow water system and thus provides the basis for the generic *dispersion relation* of the system. The dispersion relation is cubic, i.e., dependent upon ω^3 (if multiplied through by ω to eliminate the $1/\omega$ term) such that there are at most three unique solutions to equation (33a). These solutions illuminate three of our four wave types: equatorial Rossby ($n \geq 1$), mixed Rossby-gravity ($n =$

0), and inertia-gravity waves ($n \geq 1$). As Kelvin waves have no meridional structure, these solutions in *V* do not directly describe Kelvin waves; instead, they must be accounted for separately, as will be demonstrated in subsequent sections. However, it should be noted that the Kelvin wave structure can also be obtained from a consideration of (33a) for the special case where $n = -1$.

Note that the solutions described herein are presented in the case of no external forcing (e.g., heating) to drive the shallow water system. When external forcing is included, a combination of the equatorial Rossby and Kelvin wave solutions, representing the two waves with the lowest frequencies, is obtained. We will demonstrate this in a later lecture where we consider solutions to the shallow water equations for in the presence of external heat forcing. As discussed by Matsuno (1966), the lower frequency wave modes dominate the externally-forced solution because they are more responsive to a given external forcing than are the higher frequency wave modes.

From (33b), we see that there are two primary physical forcings on *V*. The first is buoyancy, analogous to *g*'*H* (akin to potential energy), wherein the equivalent depth *H* is a function of the divergence within the lower layer. The second is the meridional gradient of planetary vorticity *β*. The structure of *V* is modulated by the frequency *ω*, zonal wavenumber *k*, and the distance along the meridional axis *y*.

Physical Description of the Equatorial Wave Solutions

Solutions to equation (32) take the form of solutions to the Schrodinger equation for an oscillator (e.g., an oscillatory wave mode such as given by the shallow water system). The general form of these solutions is given by:

(34)
$$V(Y) = A\exp(-\frac{Y^2}{2})H_N(Y)$$

In equation (34), *A* is an amplitude function, $H_n(Y)$ are Hermite polynomials of order *n* and are integers and/or some multiple or power of *Y*, and *Y* is defined by:

(35)
$$Y = \left(\frac{\sqrt{g'H}}{\beta}\right)^{1/2} y$$

Note that (35) is slightly different than that contained within Appendix C of Chapter 4 in *An Introduction to Tropical Meteorology*. *Y* can be related to the Rossby radius of deformation and is defined by the ratio of buoyancy to the meridional gradient of planetary vorticity. Solutions for *V* can be found for individual values of *n*. These solutions, as noted above, can be used with (30a) to obtain solutions for H_w and, subsequently, with (27) to obtain solutions for *U*. These give the meridional amplitudes of solutions for *u'*, *v'*, and *h'*, each with wave-like structure in *x* and *t* (and a vertical structure that is unspecified herein).

As an aside, the Rossby radius of deformation L_R is typically on the order of 6000 km in the tropics. As described above, however, equatorial waves have horizontal length scales on the order of 1000-2000 km. As a result, the mass fields (such as temperature and pressure) are said to adjust in response to evolutions within the wind fields associated with the equatorial wave "perturbations" to the shallow water system. This leads to a fairly broad, relatively weak atmospheric perturbation that aligns with our expectations of linearity and finite amplitude waves inherent to the shallow water system. Indeed, the magnitude of the wind and pressure perturbations associated with each of the equatorial wave modes is relatively small (e.g., 1-2 m s^{-1} or hPa, of similar order to observational uncertainty).

At this point, we now turn to examining the solutions for each equatorial wave mode, focusing on the physical manifestations and propagation solutions inherent to each. In this manner, we will have used the shallow water system and appropriate assumptions to not only obtain dispersion relations for each wave type but to also understand their direction of propagation and how their solutions are manifest in the near-equatorial wave and pressure (i.e., mass) fields.

Equatorial Rossby Waves

Of our equatorial wave modes, equatorial Rossby waves evolve relatively slowly, similar to their mid-latitude counterparts. The period (i.e., duration) of an equatorial Rossby wave is approximately 14-21 days. As the frequency of a wave is related to the inverse of its period, equatorial Rossby waves have a relatively small frequency. This implies that ω is small. For equation (33a), small ω ($\omega << 1$) enables us to neglect the ω^2 term and enables us to write the dispersion equation for equatorial Rossby waves as:

(36) $$-\frac{\sqrt{g'H}}{\beta}\left(k^2 + \frac{\beta k}{\omega}\right) = 2n + 1$$

If we rearrange and solve for ω, we obtain:

(37) $$\omega = -\frac{\beta k}{\left(k^2 + \frac{\beta(2n+1)}{\sqrt{g'H}}\right)}$$

As with (35), note that (37) is slightly different than what is contained within Appendix C of Chapter 4 in *An Introduction to Tropical Meteorology*.

For any wave, the phase speed c_p is defined by $c_p = \omega/k$. For the dispersion relation for an equatorial Rossby wave given by (37), the phase speed is simply:

(38) $$c_p = -\frac{\beta}{\left(k^2 + \frac{\beta(2n+1)}{\sqrt{g'H}}\right)}$$

As each of the values of (38) are positive-definite, $c_p < 0$ and thus equatorial Rossby waves propagate *westward*. The speed at which they do so is determined chiefly by buoyancy and the meridional gradient in the planetary vorticity.

For n=1, the theoretical solution for equatorial Rossby waves is depicted within the lecture materials. It is characterized by westward-propagating high and low pressure centers symmetrically-displaced north and south of the equator. Wind field maxima are located along the equator. Deep, moist convection preferentially forms where convergence associated with the equatorial Rossby waves is maximized east (west) of locally lower (higher) pressures. Note that there is also weak directional convergence well to the north and south of the equator to the west (east) of locally higher (lower) pressures. However, this directional convergence is weak given the weakness of the wind fields so far from the equator and is largely mitigated by the presence of speed divergence. Deep, moist convection acts to lead to the diabatic redistribution of cyclonic potential vorticity, acting as a brake on the continued westward propagation of the downstream lower pressure centers (and thus the entire wave as a whole).

Inertia-Gravity Waves

These buoyancy-dependent waves have high frequency (i.e., large ω) and short longevity. As a result, the $-\beta k/\omega$ term in (33a) becomes negligibly small. This enables us to express the dispersion relation as the following:

(39) $$\frac{\sqrt{g'H}}{\beta}\left(\frac{\omega^2}{g'H}-k^2\right)=2n+1$$

Solving for ω by re-arranging terms, multiplying both sides by $\beta\sqrt{g'H}$, and taking the square root of both sides of the equation gives us:

(40) $$\omega=\pm\sqrt{g'Hk^2+\sqrt{g'H}\beta(2n+1)}$$

The phase speed of such waves is given by:

(41) $$c_p=\pm\frac{\sqrt{g'Hk^2+\sqrt{g'H}\beta(2n+1)}}{k}$$

We see that there are two modes of propagation for inertia-gravity waves, one eastward and one westward, corresponding to the positive and negative roots of (41) respectively.

Mixed Rossby-Gravity Waves

For the special case of mixed Rossby-gravity waves, we let n=0 in (33a) such that:

(42) $$\left(\frac{\omega^2}{g'H}-k^2-\frac{\beta k}{\omega}\right)=\frac{\beta}{\sqrt{g'H}}$$

Or, expressed in terms of ω^3,

(43) $$\frac{\omega^3}{g'H}-\omega\left(k^2+\frac{\beta}{\sqrt{g'H}}\right)-\beta k=0$$

There are three possible roots to the cubic equation given by (43). It is possible to solve for these using mathematical techniques designed to solve cubic equations; however, as this is a tedious, time-consuming process, we will instead focus on the solutions themselves. Two of these solutions (or roots) are akin to the inertia-gravity wave roots described by (40). However, the westward-moving inertia-gravity wave solution is not an allowable solution for *n*=0, reducing the number of allowable solutions to (43) from three to two. The other root of (43) gives the dispersion relation for mixed Rossby-gravity waves. For convenience, we simply write it as:

(44) $$\omega = \frac{k\sqrt{g'H}}{2}\left(1-\left(1+\frac{4\beta}{k^2\sqrt{g'H}}\right)^{1/2}\right)$$

And, as before, we can express the phase speed of these waves as:

(45) $$c_p = \frac{\sqrt{g'H}}{2}\left(1-\left(1+\frac{4\beta}{k^2\sqrt{g'H}}\right)^{1/2}\right)$$

As the $\frac{4\beta}{k^2\sqrt{g'H}}$ term in (44) and (45) is positive-definite as each of its constituents are all positive-definite themselves, $1+\frac{4\beta}{k^2\sqrt{g'H}}$ is greater than 1 such that its square root is also greater than 1. This means that the phase speed for mixed Rossby-gravity waves is negative, describing *westward* propagation of these waves.

For small *k*, the $k\sqrt{g'H}$ term in (44) is relatively small but the $\frac{4\beta}{k^2\sqrt{g'H}}$ term is relatively large. Given the influence of k^2 (rather than *k*) in the latter term, ω is relatively large and the wave behaves more like an inertia-gravity wave. For large *k*, the inverse is true, ω is relatively small, and the wave behaves more like an equatorial Rossby wave.

The theoretical solutions for a mixed Rossby-gravity wave are depicted within the lecture materials. In a dry atmosphere, the mixed Rossby-gravity wave is not tilted in the horizontal. It is characterized by clockwise and/or counterclockwise flow within ±10° latitude of the equator with maximum circulation magnitudes centered on the equator. Higher (lower) pressures are found to the north (south) of the equator for clockwise flow. Conversely, for counterclockwise flow, lower (higher) pressures are found to the north (south) of the equator. Winds are nearly meridional and at their peak magnitude along the equator; they decay rapidly to the north and south away from there, as constrained by our assumption used to obtain (33a). In a moist environment, deep, moist convection predominantly forms in regions of speed convergence. These regions are found on the eastern (western) flank of locally lower (higher) pressures. The diabatic generation of cyclonic potential vorticity in regions of convection acts as a brake on the westward movement of the wave. As the wave's pressure and convection fields are not symmetric about the equator, this also leads to the waves being tilted horizontally toward areas of speed convergence and locally lower pressures.

Kelvin Waves

As noted before, Kelvin waves have no meridional motion. Thus, equations (32) and (33a) are not exactly valid for these waves. We thus need to derive the dispersion relation for this system in another manner. To do so, we make use of the shallow water equations given by (19) and (20). These equations in *u'*, *v'*, and *h'* represent the basis of the unforced shallow water system after the bulk of our assumptions (namely the β-plane approximation and linearization) have been made to the system. If we set *v'*=0 in these equations, we obtain:

(46a) $$\frac{\partial u'}{\partial t} = -g'\frac{\partial h'}{\partial x}$$

(46b) $$\beta y u' = -g'\frac{\partial h'}{\partial y}$$

(47) $$\frac{\partial h'}{\partial t} + H\frac{\partial u'}{\partial x} = 0$$

Equations (46) and (47) give us a set of three equations for two variables (*u'* and *h'*). Now, we want to assume wave-form (wave-like) solutions for *u'* and *h'*, such as represented by equations (23a) and (23c). We substitute these solutions in to (47), making use of (24) to simplify the resultant expression:

(48) $$-i\omega H_w + HikU = 0$$

Solving for *U*, we obtain:

(49) $$U = \frac{\omega H_w}{kH}$$

If we substitute (49), (23a), and (23c) into (46a), we obtain:

(50) $$\frac{i\omega^2 H_w}{kH} = g' ikH_w$$

Solving (50) for ω, we obtain:

(51) $$\omega = \pm k\sqrt{g'H}$$

As the negative root for ω in (51) does not decay away from the equator (and, in fact, leads to amplification of the wave away from the equator), we discard it as an allowable solution. Thus, the dispersion relation for Kelvin waves is given by the positive root of (51). Note that this dispersion relation is equivalent to that for pure gravity waves. The phase speed for Kelvin waves can thus be expressed as:

(52) $$c_p = \sqrt{g'H}$$

As the radical in (52) is positive-definite, $c_p > 0$ and thus Kelvin waves propagate *eastward*. With no dependence on *k* in (52), Kelvin waves are also said to be non-dispersive.

The mathematical solution for a Kelvin wave is obtained in a similar manner to that for the other equatorial wave modes, wherein U and u' can be used to obtain H_w and h'. The resultant solutions are depicted within the lecture materials. Zonal winds are strongest along the equator and decay to the north and south from there. They are strongest at the heart (or core) of the mass/pressure field responses centered along the equator and decay east and west from there. The signs of the pressure fields can be viewed in terms of shear vorticity arguments: for westerly flow, shear vorticity to the north and south of the equator is cyclonic. Thus, along the equator, pressures should be correspondingly higher. Similarly, for easterly flow, shear vorticity is anticyclonic. Thus, along the equator, pressures should be correspondingly low. Deep, moist convection preferentially forms along the equator where convergence is maximized between pressure maxima and minima. For Kelvin waves, this is to the west (east) of lower (higher) pressure. The diabatic generation of cyclonic potential vorticity within the convection acts as a brake, as it does for the other equatorial waves, on the eastward progression of the wave.

Monitoring of Equatorial Waves

Each of the equatorial wave modes described above propagates at a unique velocity and in a unique direction. Each equatorial wave mode is also associated with a unique kinematic and, in particular, convective structure and typically lasts for a unique length of time. As a consequence, monitoring and forecasting of equatorial waves is fairly straightforward given appropriate spatial and temporal filtering of anomalous outgoing longwave radiation and upper/lower tropospheric wind fields. This allows for the isolation of a given wave mode from a set of observations or forecast fields. Specific insights into this process are provided by Wheeler and Kiladis (1999), among other references.

Unfortunately, however, there is limited skill associated with forecasts of equatorial wave activity whether such forecasts are derived from persistence (e.g., extrapolation of ongoing conditions into the future) or numerical model forecasts. Typical forecast skill extends out to 1-5 days, or up to half of a given equatorial wave's life span (or period). As a result, improving the skill of forecasts of equatorial waves and associated meteorological phenomena is an active, vibrant area of research in tropical meteorology. Section 4.1.5.2 of *An Introduction to Tropical Meteorology* contains a number of links to resources for the real-time monitoring and prediction of equatorial waves and is a recommend resource.

For Further Reading

- Chapter 4, *An Introduction to Tropical Meteorology, 2nd Edition*, A. Laing and J.-L. Evans, 2011.
- Chapter 7, An Introduction to Dynamic Meteorology, 3rd Edition, J. R. Holton, 1992.
- Gill, A. E., 1980: Some simple solutions for heat-induced tropical circulation. *Quart. J. Roy. Meteor. Soc.*, **106**, 447-462.
- Matsuno, T., 1966: Quasi-geostrophic motions in the equatorial area. *J. Meteorol. Soc. Japan*, **44**, 25–43.
- Wheeler, M., and G. N. Kiladis, 1999: Convectively coupled equatorial waves: analysis of clouds and temperature in the wavenumber-frequency domain. *J. Atmos. Sci.*, **56**, 374-399.