Synoptic Meteorology II: The Quasi-Geostrophic Vorticity Equation

10-12 February 2015

Readings: Section 2.2 of *Midlatitude Synoptic Meteorology*.

What is the Quasi-Geostrophic Approximation?

Last semester, we introduced the Rossby number:

$$Ro = \frac{U}{f_0 L} \tag{1}$$

We also introduced characteristic values of *U*, *L*, and f_0 for mid-latitude, synoptic-scale motions:

Variable	Characteristic Value	Description
u, v	$U \approx 10 \text{ m s}^{-1}$	Horizontal velocity scale
x, y	$L \approx 10^6 \text{ m}$	Length scale
f	$f_0 \approx 10^{-4} \text{ s}^{-1}$	Coriolis scale

If we plug these values in to (1), we find that the characteristic Rossby number for mid-latitude, synoptic-scale motions is 0.1. Because of how the Rossby number is defined, this defines geostrophic balance. Therefore, we state that mid-latitude, synoptic-scale motions should be at least approximately geostrophic, or *quasi-geostrophic*.

As previously stated, the total wind can be decomposed into its geostrophic and ageostrophic components:

$$\vec{\mathbf{v}} = \vec{\mathbf{v}}_g + \vec{\mathbf{v}}_{ag} \tag{2}$$

We state that the ageostrophic wind $\mathbf{v}_{ag}$ is much smaller than the geostrophic wind $\mathbf{v}_g$, or that:

$$\frac{\left\|\vec{\mathbf{v}}_{ag}\right\|}{\left\|\vec{\mathbf{v}}_g\right\|} = O(Ro) \tag{3}$$

We also restate our definition of geostrophic balance, here posed on isobaric surfaces in terms of the geopotential Φ:

$$\vec{\mathbf{v}}_g = \frac{1}{f_0}\hat{\mathbf{k}} \times \nabla_p \Phi \tag{4}$$

where the subscript *p* on the gradient operator denotes that it is being taken on a pressure surface.

You may ask, why does f_0 (and not f) appear in (4)? There are two reasons for this. First, recall that the geostrophic wind (when f is held constant) is non-divergent. We wish to make use of this property when developing the full quasi-geostrophic set of equations. Second, we find that the meridional variability in f (recalling that f is a function of latitude) is very small relative to the value of f as the meridional length scale – or north-south extent of the feature being studied – is small compared to the radius of the Earth.

Thus, we find that the geostrophic wind may be defined in terms of a constant value f_0 rather than having to include the full latitudinal variability in f. This, too, helps to simplify the complex physics of synoptic-scale weather systems without significant loss of accuracy.

<u>The Quasi-Geostrophic Primitive Equations</u>

To be able to apply principles of quasi-geostrophic theory to the study of mid-latitude, synoptic-scale weather systems, we must rewrite the full primitive equations in light of the quasi-geostrophic principles.

Horizontal Momentum Equations

The quasi-geostrophic form of the momentum equations is given by:

$$\frac{D_g \vec{\mathbf{v}}_g}{Dt} = -f_0 \hat{\mathbf{k}} \times \vec{\mathbf{v}}_{ag} - \beta y \hat{\mathbf{k}} \times \vec{\mathbf{v}}_g + \vec{\mathbf{F}} \qquad (5)$$

where $f = f_0 + \beta y$, an approximation known as the mid-latitude beta plane approximation. Here, $f_0 = f$ at some reference latitude ϕ_0, $\beta = (df/dy)$ evaluated at ϕ_0, and $y = 0$ at ϕ_0. **F** represents friction and is often neglected, a legitimate assumption above the planetary boundary layer.

The total derivative on the left-hand side of (5) takes the general form:

$$\frac{D_g}{Dt}(\ ) \equiv \frac{\partial}{\partial t}(\ ) + \vec{\mathbf{v}}_g \cdot \nabla(\ ) \equiv \frac{\partial}{\partial t}(\ ) + u_g \frac{\partial}{\partial x}(\ ) + v_g \frac{\partial}{\partial y}(\ )$$

Neglecting friction, the terms on the right-hand side of (5) are both Coriolis terms, and thus accelerations of the geostrophic flow are driven by Coriolis forcing. In simplified terms, the first represents the multiplication of f_0 with the ageostrophic wind, whereas the second represents the multiplication of the latitudinal variability of f with the geostrophic wind.

Because of the relative magnitudes of $\mathbf{v}_{ag}$ and βy as compared to $\mathbf{v}_g$ and f_0, respectively, both terms are one order of magnitude smaller than the product of f_0 and $\mathbf{v}_g$, which itself is manifest as the Coriolis force. Since the Coriolis force is in balance with the pressure gradient force under conditions of geostrophic balance, each of the right-hand side terms in (5) is approximately one order of magnitude smaller than the magnitude of the pressure gradient force.

Vertical Momentum Equation

The quasi-geostrophic form of the vertical momentum equation is given by the hydrostatic equation:

$$\frac{\partial \Phi}{\partial p} = -\frac{RT}{p} \tag{6}$$

To best interpret (6) in the context of the quasi-geostrophic approximation, it is useful to recall precisely what is meant by the concept of hydrostatic balance. Nominally, hydrostatic balance is defined *in the absence of atmospheric motions* as the balance between the gravitational force and the vertical component of the pressure gradient force. Conceptually, by invoking hydrostatic balance, we are implicitly stating that synoptic-scale vertical motions are of small magnitude.

Continuity Equation

The quasi-geostrophic form of the continuity equation is given by:

$$\nabla \cdot \vec{\mathbf{v}}_{ag} + \frac{\partial \omega}{\partial p} = 0 \tag{7}$$

Why does only the ageostrophic wind appear in (6)? We previously demonstrated the geostrophic wind to be non-divergent (i.e., $\nabla \cdot \vec{\mathbf{v}}_g = 0$). Thus, upon substituting (2) for the full wind in the continuity equation, only the ageostrophic component remains:

$$\nabla \cdot \vec{\mathbf{v}} = \nabla \cdot \left(\vec{\mathbf{v}}_g + \vec{\mathbf{v}}_{ag}\right) = \nabla \cdot \vec{\mathbf{v}}_g + \nabla \cdot \vec{\mathbf{v}}_{ag} = \nabla \cdot \vec{\mathbf{v}}_{ag}$$

The principle deduction that we can make from (7) is fairly straightforward: synoptic-scale vertical motion is determined by the ageostrophic component of the wind field alone. We will return to this principle at times throughout the course.

Thermodynamic Equation

Finally, the quasi-geostrophic form of the thermodynamic equation is given by:

$$\frac{\partial T}{\partial t} + \vec{\mathbf{v}}_g \cdot \nabla T - S_p \omega = \frac{1}{c_p} \frac{dQ}{dt} \tag{8}$$

where S_p is the static stability (which can be approximated by $-\partial \theta / \partial p$) and dQ/dt is the diabatic heating rate (e.g., radiative input or loss, moisture phase changes, etc.).

From left to right in (8), the terms represent the local time rate of change of temperature, the (horizontal) advection of temperature by the geostrophic wind, adiabatic warming/cooling, and diabatic warming/cooling. The primary difference between the full and quasi-geostrophic thermodynamic equations is the presence of the geostrophic (rather than the full) wind in the horizontal advection term of the latter.

The adiabatic warming/cooling term, $S_p\omega$, reflects temperature changes associated with vertical motions under adiabatic (i.e., potential temperature-conserving) conditions. Nominally, we presume a statically-stable atmosphere, such that S_p is positive-definite. As a parcel ascends ($\omega < 0$) dry adiabatically, its temperature cools ($\partial T/\partial t < 0$) due to adiabatic expansion. As a parcel descends ($\omega > 0$) dry adiabatically, its temperature warms ($\partial T/\partial t > 0$) due to adiabatic compression. Even though vertical motion is typically small on the synoptic-scale, this term cannot be neglected because the static stability is typically comparatively large.

Defining the Geostrophic Relative Vorticity

Earlier, we stated that the geostrophic wind on isobaric surfaces takes the form:

$$\vec{\mathbf{v}}_g = \frac{1}{f_0}\hat{\mathbf{k}} \times \nabla_p \Phi \tag{9}$$

where Φ is the geopotential and the subscript of p denotes that the gradient operator is applied on a constant pressure surfaces. If we expand (9) into its components, for u_g and v_g, we obtain:

$$u_g = -\frac{1}{f_0}\frac{\partial \Phi}{\partial y} \tag{10a}$$

$$v_g = \frac{1}{f_0}\frac{\partial \Phi}{\partial x} \tag{10b}$$

Recall that the relative vorticity, or ζ, is given by:

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \tag{11}$$

The geostrophic form of the relative vorticity can be expressed similarly as:

$$\zeta_g = \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} \tag{12}$$

Because we have definitions for both u_g and v_g, as given by (10), we can go a step further than this. Plugging (10) into (12), we obtain:

$$\zeta_g = \frac{\partial}{\partial x}\left(\frac{1}{f_0}\frac{\partial \Phi}{\partial x}\right) - \frac{\partial}{\partial y}\left(-\frac{1}{f_0}\frac{\partial \Phi}{\partial y}\right) \qquad (13)$$

Because f_0 is a constant with respect to both x and y, the factors of $1/f_0$ can be pulled out of each derivative. Therefore, (13) simplifies to:

$$\zeta_g = \frac{1}{f_0}\left(\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2}\right) = \frac{1}{f_0}\nabla_p^2 \Phi \qquad (14)$$

In (14), ∇^2 is known as the Laplacian operator. As with (9), the subscript p denotes that it is computed on an isobaric surface.

Before we proceed, a brief digression about the meaning of (14). It denotes that if you know the geopotential field on a given pressure surface, under the constraint of geostrophic balance, you can obtain the geostrophic relative vorticity and the geostrophic winds on that pressure surface. Or, conversely, if you know the geostrophic winds on a given pressure surface, you can obtain the geopotential field on that given surface. This concept, known as *invertability*, is something that we'll explore further (as time permits) when we discuss potential vorticity.

Furthermore, (14) also gives us insight into the distribution of geostrophic relative vorticity with respect to the geopotential field. The Laplacian operator ∇^2 can be viewed as a measurement of the curvature of a field. Nominally, where $\nabla^2\Phi$ is a maximum, Φ itself is a minimum (and vice versa). As a result, maxima in $\nabla^2\Phi$ (and thus also ζ_g) correspond to minima in Φ. Likewise, minima in $\nabla^2\Phi$ (and thus also ζ_g) correspond to maxima in Φ.

Since $\Phi = gz$ and g is a constant, these assessments can be interpreted in terms of the height field rather than just the geopotential…

- Geostrophic relative vorticity is maximized (or is cyclonic) at the base of a trough on a given pressure surface.

- Geostrophic relative vorticity is minimized (or is anticyclonic) at the apex of a ridge on a given pressure surface.

Why are we interested in the geostrophic relative vorticity? The evolution of geostrophic relative vorticity, both in space as well as in time, provides vital information that can be used to diagnose the movement of synoptic-scale meteorological phenomena. Likewise, it provides vital information that can be used to diagnose synoptic-scale vertical motions associated with such phenomena. In the aggregate, such information can, in part, be used to describe the formation, motion, and evolution of mid-latitude cyclones. Much of our discussion of quasi-geostrophic theory follows directly from these tenets, as we will explore in upcoming lectures.

The Quasi-Geostrophic Vorticity Equation

Similar to the computation of relative vorticity, we obtain the quasi-geostrophic vorticity equation by finding *∂/∂x* of the *y* (or **j**) component of the quasi-geostrophic momentum equation and subtracting from it *∂/∂y* of the *x* (or **i**) component of the quasi-geostrophic momentum equation.

Neglecting friction, recall that the vector form of the quasi-geostrophic horizontal momentum equation takes the form:

$$\frac{D_g \vec{\mathbf{v}}_g}{Dt} = -f_0 \hat{\mathbf{k}} \times \vec{\mathbf{v}}_{ag} - \beta y \hat{\mathbf{k}} \times \vec{\mathbf{v}}_g \tag{15}$$

Expanded into its components, (15) becomes:

$$\frac{D_g u_g}{Dt} = f_0 v_{ag} + \beta y v_g \tag{16a}$$

$$\frac{D_g v_g}{Dt} = -f_0 u_{ag} - \beta y u_g \tag{16b}$$

To form the quasi-geostrophic vorticity equation, we thus compute *∂/∂x* of (16b) - *∂/∂y* of (16a):

$$\frac{\partial}{\partial x}\left(\frac{D_g v_g}{Dt}\right) - \frac{\partial}{\partial y}\left(\frac{D_g u_g}{Dt}\right) = \frac{D_g}{Dt}\left(\frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y}\right) = \frac{D_g \zeta_g}{Dt} \tag{17a}$$

$$\frac{D_g \zeta_g}{Dt} = \frac{\partial}{\partial x}\left(-f_0 u_{ag} - \beta y u_g\right) - \frac{\partial}{\partial y}\left(f_0 v_{ag} + \beta y v_g\right) \tag{17b}$$

(17a) reflects the left-hand side of the combined equation while (17b) reflects the final result. Expanding (17b) and subsequently grouping like terms, we obtain:

$$\begin{aligned}\frac{D_g \zeta_g}{Dt} &= -f_0 \frac{\partial u_{ag}}{\partial x} - f_0 \frac{\partial v_{ag}}{\partial y} - \beta y \frac{\partial u_g}{\partial x} - \beta y \frac{\partial v_g}{\partial y} - \beta v_g \\ &= -f_0\left(\frac{\partial u_{ag}}{\partial x} + \frac{\partial v_{ag}}{\partial y}\right) - \beta y\left(\frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y}\right) - \beta v_g\end{aligned} \tag{18}$$

Because the divergence of the geostrophic wind, when f is held constant ($f = f_0$), is equal to zero, the $-\beta y$ term in (18) will vanish. Likewise, the continuity equation (7) can be used to rewrite the terms involving u_{ag} and v_{ag} in terms of the vertical motion (or, more specifically, the vertical derivative thereof). With these concepts in mind, (18) becomes:

$$\frac{D_g \zeta_g}{Dt} = f_0 \frac{\partial \omega}{\partial p} - \beta v_g \tag{19}$$

Likewise, by plugging in ζ_g into the definition of the total derivative, the total derivative on the left-hand side of (19) can be expressed as:

$$\frac{D_g \zeta_g}{Dt} = \frac{\partial \zeta_g}{\partial t} + \vec{\mathbf{v}}_g \cdot \nabla\left(\zeta_g\right) \tag{20}$$

Using (20), (19) may be written as:

$$\frac{\partial \zeta_g}{\partial t} = -\vec{\mathbf{v}}_g \cdot \nabla \zeta_g - \beta v_g + f_0 \frac{\partial \omega}{\partial p} \tag{21}$$

Equation (21) is the *quasi-geostrophic vorticity equation*. It is a *prognostic* equation for the geostrophic relative vorticity. It states that the local change of geostrophic relative vorticity is a function of three terms. Expressed from left to right, these terms are (a) the geostrophic horizontal advection of geostrophic relative vorticity, (b) the geostrophic meridional advection of planetary vorticity, and (c) the vertical stretching of planetary vorticity.

Application of the Quasi-Geostrophic Vorticity Equation

Advection Processes

To begin our interpretation of (21), we wish to consider only the first two terms on its right-hand side, those related to horizontal advection of geostrophic relative vorticity and planetary vorticity. To do so, we will utilize a schematic of an idealized synoptic-scale trough/ridge pattern, as given by Figure 1 below.

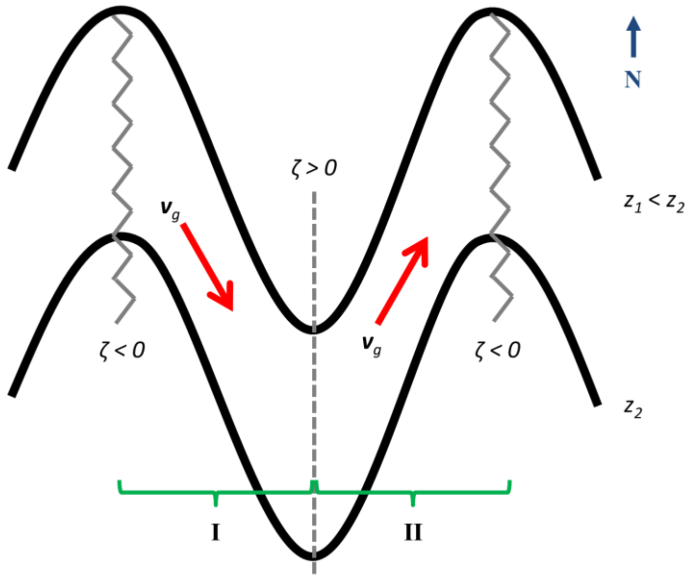

Figure 1. Schematic 500 hPa geopotential height field showing regions of positive and negative geostrophic horizontal advections of planetary and geostrophic relative vorticity.

To the east of the base of the trough (region II in Figure 1), the meridional flow is from south to north ($v_g > 0$). Because f increases with increasing latitude in the Northern Hemisphere, and because β is defined as the derivative of f with respect to latitude (i.e., $\beta = \partial f/\partial y$), β is positive. With the leading negative sign, the planetary advection term is negative, implying a decrease in geostrophic relative vorticity. However, the large-scale flow is directed from the base of the trough, where the geostrophic relative vorticity is cyclonic (positive-definite in the northern hemisphere). This implies an increase in geostrophic relative vorticity, and thus the two advection terms are of opposing sign.

Conversely, to the west of the base of the trough (region I in Figure 1), the meridional flow is from north to south ($v_g < 0$). β remains positive, such that with the leading negative sign, the planetary advection term is positive, implying an increase in geostrophic relative vorticity. Likewise, the large-scale flow is directed from the apex of the ridge, where the geostrophic relative vorticity is anticyclonic (negative-definite in the northern hemisphere). This implies a decrease in geostrophic relative vorticity, and thus the two advection terms are again of opposing sign.

The two advection terms can be used to describe only the ***motion*** – and **not** the *amplification or deamplification* – of the synoptic-scale, mid-latitude trough/ridge pattern. Why is this? First, recall that the geostrophic wind blows parallel to the contours of constant geopotential height. For a trough or ridge to amplify or deamplify, the geostrophic wind would need to have a non-zero component blowing perpendicular to the contours of constant geopotential height. Since it does not, particularly in the base of a trough or apex of a ridge, it cannot change the amplitude of the pattern. Second, recall that the geopotential height field and geostrophic relative vorticity are inextricably linked through (14). If the geostrophic relative vorticity is advected from one location to another by the geostrophic wind, the geopotential height field and the geostrophic wind itself will change to match.

To better illustrate this, let us examine this term in the base of the trough depicted in Figure 1. The geostrophic wind here is directed from west to east, parallel to the geopotential height contours. Here, the β term in (21) is zero because v_g is zero. The geostrophic relative vorticity advection term acts to advect (or transport) the cyclonic geostrophic relative vorticity maximum to the east. As this happens, however, the geopotential height field will shift to the east as well so that the geopotential height minimum remains where the geostrophic relative vorticity maximum is located. The geostrophic wind remains easterly, and the process continues. The same arguments can be made in the apex of one of the ridges in Figure 1.

How do we know which of the two advection terms is of larger magnitude? While the derivation necessary to understand this is beyond the scope of this class, the results of doing so indicate:

- For shortwave troughs, or those of zonal (east-west) extent less than approximately 3,000 km, the geostrophic horizontal advection of geostrophic relative vorticity dominates over the geostrophic meridional advection of planetary vorticity. Thus, these features generally move eastward with the westerly large-scale flow.

- For longwave troughs, or those of zonal extent greater than approximately 10,000 km, the geostrophic meridional advection of planetary vorticity dominates over the geostrophic horizontal advection of geostrophic relative vorticity. Thus, to first approximation, these features move westward, or *retrogress*, against the westerly large-scale flow.

- Troughs of intermediate zonal extent (between 3,000-10,000 km) tend to move eastward, albeit at a rate of speed slower to much slower than that of the westerly large-scale flow.

In observations, however, longwave troughs tend to remain relatively stationary rather than retrogress. There are many reasons why this may be the case, many of which are not explicitly considered in the context of quasi-geostrophic theory: topographic influences, large-scale thermal contrasts (e.g., land versus water), and non-linear interaction(s) with shortwave troughs rotating around the base of the longwave trough. Thus, it is helpful to remember that quasi-geostrophic theory is merely an *approximation* to the real atmosphere!

Inclusion of Vertical Motion

In the above, we neglected the third term on the right-hand side of (21), or that associated with the vertical stretching of planetary vorticity. This process can be viewed in the context of a figure skater performing a spin on ice. As the skater begins their spin, they are low to the ground. Their rate of rotation is relatively slow (or small). However, as they continue their spin, they bring (or converge) their arms inward, becoming more upright – or stretched vertically – as they do so. This intensifies their rate of rotation.

Applying this concept to (21), consider the case where there is *rising* motion throughout the troposphere that is maximized within the middle troposphere (such as near 500 hPa). Because $\omega < 0$ denotes rising motion, $\partial\omega$ is negative in the lower to middle troposphere. Likewise, ∂p is negative because pressure decreases with increasing altitude. Thus, the last term on the right-hand side of (21) is positive. This implies an increase in the local magnitude of the geostrophic relative vorticity.

The analogy and application work in the inverse, as well. Consider the case where there is *descending* motion throughout the troposphere that is maximized within the middle troposphere. Because $\omega > 0$ denotes descending motion, $\partial\omega$ is positive in the lower to middle troposphere. As before, ∂p is negative because pressure decreases with increasing altitude. Thus, the last term on the right-hand side of (21) is negative. This implies a decrease in the local magnitude of the geostrophic relative vorticity as the vortex tube (or ice skater) is compressed toward the ground.

Thus, when considering the local evolution of geostrophic relative vorticity, both horizontal advection as well as vertical motions must be considered! From the continuity equation, these vertical motions are exclusively a function of the ageostrophic wind, which in and of itself is tied to parcel accelerations. As we study the evolution of synoptic-scale weather systems, we will see precisely how horizontal advection and vertical motions work in concert with one another to influence the evolutions of these phenomena.