Synoptic Meteorology II: The Quasi-Geostrophic Omega Equation

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Readings: Sections 2.3 and 2.5 of *Midlatitude Synoptic Meteorology*.

Why Are We Interested In Vertical Motion?

Before we get into the derivation and discussion of the quasi-geostrophic omega equation, it is prudent for us to ask: why, precisely, are we interested in vertical motion? Firstly, recall our discussion of the quasi-geostrophic vorticity equation. We found that the $\partial\omega/\partial p$ term contained within that equation is responsible for the amplification and/or deamplification of middle tropospheric troughs and ridges. As a result, in the quasi-geostrophic system, if we want to know something about how the amplitude of the mid-latitude trough/ridge pattern is evolving, we need to know how the vertical velocity varies with respect to pressure.

Secondly, recall that upon dry adiabatic ascent, mixing ratio is conserved. Thus, ascent over a deep enough vertical layer or prolonged period of time brings about condensation. This is substantially aided if the layer in which the ascent occurs is relatively moist prior to the ascent beginning. Naturally, the overlap of ascent and moisture implies cloud and precipitation formation. As a result, we are interested in vertical motion because, even on the synoptic-scale where vertical motions are normally weak, it plays a role in the formation and evolution of clouds and precipitation.

Obtaining The Quasi-Geostrophic Omega Equation

Recall that the quasi-geostrophic vorticity equation is given by the following:

$$\nabla^2\left(\frac{\partial\Phi}{\partial t}\right) = f_0\left(-\vec{\mathbf{v}}_g\cdot\nabla\left(\zeta_g + f\right)\right) + f_0^2\frac{\partial\omega}{\partial p} - f_0 K\zeta_g \tag{1}$$

Likewise, recall that the quasi-geostrophic thermodynamic equation is given by the following:

$$-\frac{\partial}{\partial p}\left(\frac{\partial\Phi}{\partial t}\right) = -\vec{\mathbf{v}}_g\cdot\nabla\left(h\theta\right) + \sigma\omega + \frac{R}{pc_p}\frac{dQ}{dt} \tag{2}$$

The quasi-geostrophic vorticity (5) and thermodynamic (13) equations represent two equations containing two unknowns – vertical motion ω and geopotential height Φ – or variables such as the geostrophic relative vorticity ζ_g and potential temperature θ that can be obtained from knowledge of the geopotential height field. To obtain the quasi-geostrophic omega equation, we wish to combine these two equations in a way that eliminates Φ between the two equations, leaving a single equation for ω that describes the vertical motion on a given isobaric surface.

To do so, we need to obtain $\partial/\partial p$ of (1). Doing so, and commuting the derivatives on the left-hand side of the resultant equation, we obtain:

$$\nabla^2\left(\frac{\partial}{\partial p}\frac{\partial\Phi}{\partial t}\right)=f_0\frac{\partial}{\partial p}\left(-\vec{\mathbf{v}}_g\cdot\nabla\left(\zeta_g+f\right)\right)+f_0^2\frac{\partial^2\omega}{\partial p^2}-f_0\frac{\partial}{\partial p}\left(K\zeta_g\right) \tag{3}$$

Likewise, we need to obtain ∇^2 of (2). Doing so, and slightly re-writing the first term on the right-hand side of the resultant equation, we obtain:

$$-\nabla^2\left(\frac{\partial}{\partial p}\frac{\partial\Phi}{\partial t}\right)=h\nabla^2\left(-\vec{\mathbf{v}}_g\cdot\nabla\theta\right)+\sigma\nabla^2\omega+\frac{R}{pc_p}\nabla^2\left(\frac{dQ}{dt}\right) \tag{4}$$

If we add (3) and (4) and bring the terms involving ω to the left-hand side of the resultant equation, we obtain the *quasi-geostrophic omega equation*:

$$\sigma\nabla^2\omega+f_0^2\frac{\partial^2\omega}{\partial p^2}=-f_0\frac{\partial}{\partial p}\left(-\vec{\mathbf{v}}_g\cdot\nabla\left(\zeta_g+f\right)\right)-h\nabla^2\left(-\vec{\mathbf{v}}_g\cdot\nabla\theta\right)+f_0\frac{\partial}{\partial p}\left(K\zeta_g\right)-\frac{R}{pc_p}\nabla^2\left(\frac{dQ}{dt}\right) \tag{5}$$

Equation (5) is a partial differential equation describing the vertical motion ω on an isobaric surface. There are four forcing terms on the right-hand side of (5). From left to right, these represent differential geostrophic vorticity advection, the Laplacian of potential temperature advection, differential friction, and the Laplacian of diabatic heating.

As with the quasi-geostrophic height tendency equation, this equation is typically applied in the middle troposphere and not at the surface. When combined with the quasi-geostrophic vorticity equation (1), however, the evolution of synoptic-scale weather features – including surface pressure systems – can be examined. We will tackle this in a future lecture.

The left-hand side of (5) expresses ω in terms of the Laplacian (∇^2) as well as a second derivative with respect to pressure. Based upon the definitions of these operators, the left-hand side of (5) can be approximated as:

$$\sigma\nabla^2\omega+f_0^2\frac{\partial^2\omega}{\partial p^2}\propto-\omega \tag{6}$$

As before, the $\propto$ symbol means "is proportional to," such that the left-hand side of (6) is proportional to $-\omega$. Therefore, where the right-hand side of (5) is positive, ω is negative. Because of the sign convention on ω, this implies local *ascent*. Likewise, where the right-hand side of (5) is negative, ω is positive, implying local *descent*.

Basic Interpretation of the Quasi-Geostrophic Omega Equation

Differential Geostrophic Vorticity Advection

The contribution to vertical motion exclusively due to differential geostrophic vorticity advection can be expressed by:

$$\omega \propto f_0 \frac{\partial}{\partial p}\left(-\vec{\mathbf{v}}_g \cdot \nabla\left(\zeta_g + f\right)\right) \tag{7}$$

To interpret (7), we consider the cases where (a) cyclonic geostrophic vorticity advection increases with increasing height and (b) anticyclonic geostrophic vorticity advection increases with increasing height. The general sign convention on the advection terms was discussed in the previous lecture on the quasi-geostrophic height tendency equation.

In case (a), the numerator on the right-hand side of (7) is positive. The denominator, the change in pressure with height, is negative – as it always is, because pressure decreases with increasing height. Therefore, ω is negative, which implies middle tropospheric *ascent*. In case (b), the numerator on the right-hand side of (7) is negative. The denominator here is also negative. Therefore, ω is positive, which implies middle tropospheric *descent*.

Laplacian of Potential Temperature Advection

The contribution to vertical motion exclusively due to the Laplacian of potential temperature advection can be expressed by:

$$\omega \propto h\nabla^2\left(-\vec{\mathbf{v}}_g \cdot \nabla\theta\right) \tag{8}$$

The right-hand side of (8) includes a Laplacian operator, which is difficult to readily interpret. To simplify, we make use of a generalized form of the relationship posed by (6), such that:

$$\omega \propto -h\left(-\vec{\mathbf{v}}_g \cdot \nabla\theta\right) \tag{9}$$

Note that the specific definition of *h*, which is positive-definite, is provided within the quasi-geostrophic height tendency equation lecture notes. Instead, we focus on the geostrophic potential temperature advection term within the parentheses on the right-hand side of (9). Per the leading negative on the right-hand side of (9), warm geostrophic potential temperature advection on a given isobaric surface results in $\omega < 0$ (or ascent). Likewise, cold geostrophic potential temperature advection on a given isobaric surface results in $\omega > 0$ (or descent).

One interesting digression before proceeding: a hallmark of a baroclinic atmosphere is the presence of horizontal temperature (or potential temperature) gradients. The *baroclinicity* can be viewed as a measure of the strength of those gradients, or how rapidly the temperature changes over a given distance. Because of the proportionality expressed in (9), we can state that the magnitude of ω is proportional to the magnitude of the baroclinicity (or temperature gradient) of the synoptic-scale environment.

Differential Friction

The contribution to vertical motion exclusively due to differential friction can be expressed by:

$$\omega \propto -f_0 \frac{\partial}{\partial p}\left(K\zeta_g\right) \tag{10}$$

In (10), K represents the effects of friction and is positive-definite. K is non-zero only within the boundary layer, close to the surface, where the frictional effects of the land-surface can be meaningfully communicated to the troposphere.

The right-hand side of (10) contains a partial derivative with respect to pressure. However, because K is non-zero only within the boundary layer, the product $K\zeta_g$ will be zero in the middle to upper troposphere. As a result, the sign of the right-hand side of (10) depends entirely upon the sign of the geostrophic relative vorticity ζ_g within the lower troposphere.

For the case where cyclonic geostrophic relative vorticity ($\zeta_g > 0$) is found within the lower troposphere, the numerator on the right-hand side of (10) will be negative. The denominator, the change of pressure with height, is again negative. Per the leading negative on the right-hand side of (10), this implies $\omega < 0$, or *ascent*. This is what is known as *Ekman pumping*.

For the case where anticyclonic geostrophic relative vorticity ($\zeta_g < 0$) is found within the lower troposphere, the numerator on the right-hand side of (10) will be positive. The denominator, the change of pressure with height, is negative. Per the leading negative on the right-hand side of (10), this implies $\omega > 0$, or *descent*. This is what is known as *Ekman suction*.

Laplacian of Diabatic Heating

The contribution to vertical motion exclusively due to the Laplacian of diabatic heating can be expressed by:

$$\omega \propto \frac{R}{pc_p}\nabla^2\left(\frac{dQ}{dt}\right) \tag{11}$$

dQ/dt is the diabatic heating rate. Diabatic warming refers to the situation where $dQ/dt > 0$, while diabatic cooling refers to the situation where $dQ/dt < 0$. This term is non-zero only in the presence of diabatic heating, such as from radiation and latent heat release, nominally in a saturated atmosphere. Oftentimes, on the synoptic-scale where motions are primarily adiabatic in nature and the atmosphere is unsaturated, this term is neglected.

As with the Laplacian of potential temperature advection, the right-hand side of (11) includes a Laplacian operator that is difficult to readily interpret. To simplify, we make use of a generalized form of the relationship posed by (6), such that:

$$\omega \propto -\frac{R}{pc_p}\frac{dQ}{dt} \tag{12}$$

Thus, the presence of diabatic warming leads to $\omega < 0$, implying *ascent*. The presence of diabatic cooling leads to $\omega > 0$, implying *descent*.

<u>The Quasi-Geostrophic Omega Equation and Geostrophic Balance</u>

Preliminary Considerations

For convenience, let us restate the quasi-geostrophic omega equation:

$$\sigma\nabla^2\omega + f_0^2\frac{\partial^2\omega}{\partial p^2} = -f_0\frac{\partial}{\partial p}\left(-\vec{\mathbf{v}}_g\cdot\nabla\left(\zeta_g + f\right)\right) - h\nabla^2\left(-\vec{\mathbf{v}}_g\cdot\nabla\theta\right) + f_0\frac{\partial}{\partial p}\left(K\zeta_g\right) - \frac{R}{pc_p}\nabla^2\left(\frac{dQ}{dt}\right) \tag{13}$$

Assuming that diabatic heating is absent or can be estimated in some fashion, the right-hand side of (13) involves only geostrophic quantities. Obviously, the geostrophic wind $\boldsymbol{v}_g$ and the geostrophic relative vorticity ζ_g are geostrophic quantities. However, through use of Poisson's relationship, the hydrostatic equation, and the definition of geostrophic relative vorticity in terms of the geopotential, the potential temperature θ can also be viewed as a geostrophic quantity.

As a result, synoptic-scale vertical motions – weak as they may be – are driven entirely by the geostrophic flow. Let us consider this thought in the context of the continuity equation, however:

$$\nabla \cdot \vec{\mathbf{v}}_{ag} + \frac{\partial \omega}{\partial p} = 0 \tag{14}$$

The continuity equation states that the vertical motion is intricately tied to the divergent component (not the entirety) of the ageostrophic wind. Because the geostrophic flow exclusively provides forcing the vertical motions, (14) implies that it provides forcing for the divergent component of the ageostrophic wind. As the ageostrophic wind inherently implies a departure from geostrophic balance, *this means that the geostrophic flow is responsible for bringing departures from geostrophic balance!*

The ageostrophic circulation – that comprised of the ageostrophic wind and vertical motion – acts to restore the geostrophic (and hydrostatic) balance that the geostrophic wind itself destroyed. In the quasi-geostrophic system, this can be viewed in the context of quasi-geostrophic vorticity equation:

$$\frac{\partial \zeta_g}{\partial t} = -\vec{\mathbf{v}}_g \cdot \nabla \zeta_g - \beta v_g + f_0 \frac{\partial \omega}{\partial p} \tag{15}$$

The last term on the right-hand side of (15), involving the partial derivative of ω with respect to pressure, is the manifestation of how the ageostrophic circulation acts to restore geostrophic balance. The ageostrophic flow impacts the geostrophic relative vorticity and, because of its definition, the geopotential height and related fields.

In the following discussion, we will demonstrate these concepts in the context of each of the forcing terms to the quasi-geostrophic omega equation. Before we do so, however, it is helpful to introduce a couple of terms that you may encounter in this class or in your own readings:

- *Primary circulation* – the horizontal geostrophic flow.
- *Secondary circulation* – the ageostrophic circulation, comprised to first order of the vertical motion and the divergent component of the ageostrophic flow.

Application to the Quasi-Geostrophic Omega Equation

Differential Geostrophic Vorticity Advection

Let us consider the case where cyclonic geostrophic vorticity advection is increasing with increasing height, thus forcing ascent per our discussion above. The continuity equation, given by (14), relates the partial derivative of the vertical motion ω with respect to pressure to the divergence, which in the quasi-geostrophic system is entirely driven by the ageostrophic flow.

In this discussion, we assume that the ascent is maximized in the middle troposphere, where we have applied the quasi-geostrophic omega equation, and decays to zero at both the rigid boundaries of the surface and tropopause. As a result, $\partial\omega/\partial p$ in the lower troposphere is positive and $\partial\omega/\partial p$ in the upper troposphere is negative. From (14), we know that the former is associated with *convergence* and the latter with *divergence*.

Next, consider the quasi-geostrophic vorticity equation, given by (15) above. For simplicity, let us only consider the forcing due to the vertical motion term, i.e.,

$$\frac{\partial \zeta_g}{\partial t} = f_0 \frac{\partial \omega}{\partial p} \tag{16}$$

In the lower troposphere, where $\partial\omega/\partial p$ is positive, so too is $\partial\zeta_g/\partial t$. This implies that middle tropospheric ascent forces increasing geostrophic relative vorticity with time in the lower troposphere! In the upper troposphere, where $\partial\omega/\partial p$ is negative, so too is $\partial\zeta_g/\partial t$. This implies that middle tropospheric ascent forces decreasing geostrophic relative vorticity with time in the upper troposphere! In other words, *ascent counteracts the initial situation of cyclonic vorticity advection increasing with height that forced the vertical motion in the first place!*

Similar arguments can be made to describe the case where anticyclonic vorticity advection increases with increasing height. The geostrophic flow forces synoptic-scale descent; the ageostrophic circulation associated with such ascent concurrently counteracts the initial situation of anticyclonic vorticity advection increasing with height.

Potential Temperature Advection

Let us consider the case where there is warm potential temperature advection on the isobaric level on which the quasi-geostrophic omega equation is applied, again associated with ascent. Due to the definition of potential temperature, because pressure is constant on an isobaric surface, warm potential temperature advection implies warm temperature advection.

In the absence of diabatic processes, however, dry adiabatic ascent results in cooling (of temperature, though not potential temperature, which is conserved under such conditions). This can be viewed in the context of ascent along a dry adiabatic on a skew-T diagram. As a result, *ascent counteracts the initial situation of warm (potential) temperature advection that forced the vertical motion in the first place!* Similar arguments can be made to describe the response to cold potential temperature advection.

Differential Friction

Let us consider the case where there is cyclonic geostrophic relative vorticity in the boundary layer, again associated with ascent. As noted above in our discussion of balance in the context of the differential geostrophic vorticity advection term, this results in $\partial\omega/\partial p > 0$ in the lower troposphere and $\partial\omega/\partial p < 0$ in the upper troposphere. Likewise, as before, this results in $\partial\zeta_g/\partial t > 0$ in the lower troposphere and $\partial\zeta_g/\partial t < 0$ in the upper troposphere. As a result, *ascent counteracts the initial situation of friction acting to dissipate cyclonic vorticity in the lower troposphere that forced the vertical motion in the first place!* Similar arguments can be made for the case where friction acts on anticyclonic geostrophic relative vorticity within the boundary layer.

Diabatic Heating

Finally, let us consider the case where there is diabatic warming (i.e., $dQ/dt > 0$). The physical interpretation is identical to that for warm potential temperature advection, as stated above. The ascent-driven adiabatic cooling that results from diabatic warming counteracts the warming. Similar arguments can be made in the inverse for diabatic cooling.

The Quasi-Geostrophic Omega Equation Applied to the Four Quadrant Jet Model

One can also invoke the quasi-geostrophic omega equation to interpret regions of ascent and descent within the four quadrant jet model. If we consider only the differential geostrophic absolute vorticity forcing term, the quasi-geostrophic omega equation can be approximated by:

$$\omega \propto f_0 \frac{\partial}{\partial p}\left(-\vec{\mathbf{v}}_g \cdot \nabla\left(\zeta_g + f\right)\right) \tag{17}$$

Making this approximation implies that the other terms of the quasi-geostrophic omega equation are negligible (i.e., no potential temperature advection, friction, or diabatic heating).

Poleward of a westerly jet streak, there is typically an upper tropospheric trough. Cyclonic geostrophic absolute vorticity is maximized in the base of this trough, as we discussed earlier in the semester. Equatorward of a westerly jet streak, there is typically an upper tropospheric ridge. Anticyclonic geostrophic absolute vorticity is maximized in the apex of this ridge. This configuration results in upper tropospheric cyclonic vorticity advection in the left exit and right entrance regions of the jet streak and upper tropospheric anticyclonic vorticity advection in the right exit and left entrance regions of the jet streak. If we presume that vorticity advection is small near the surface, this indicates that there should be middle tropospheric ascent in the right entrance and left exit regions of the jet streak and middle tropospheric descent in the left entrance and right exit regions of the jet streak.

If we consider only the potential temperature advection forcing term, the quasi-geostrophic omega equation can be approximated by:

$$\omega \propto -h\left(-\vec{\mathbf{v}}_g \cdot \nabla\theta\right) \tag{18}$$

Making this approximation implies that the other terms of the quasi-geostrophic omega equation are negligible (i.e., no differential geostrophic relative vorticity advection, friction, or diabatic heating).

We wish to consider two cases: one where the isentropes (or, equivalently on an isobaric surface, isotherms) are oriented such that there is cold advection through the jet streak and one where the isentropes or isotherms are oriented such that there is warm advection through the jet streak. These are depicted below in Figure 1(b) and 1(c), respectively. For reference, the case with no potential temperature advection through the jet streak is depicted below in Figure 1(a).

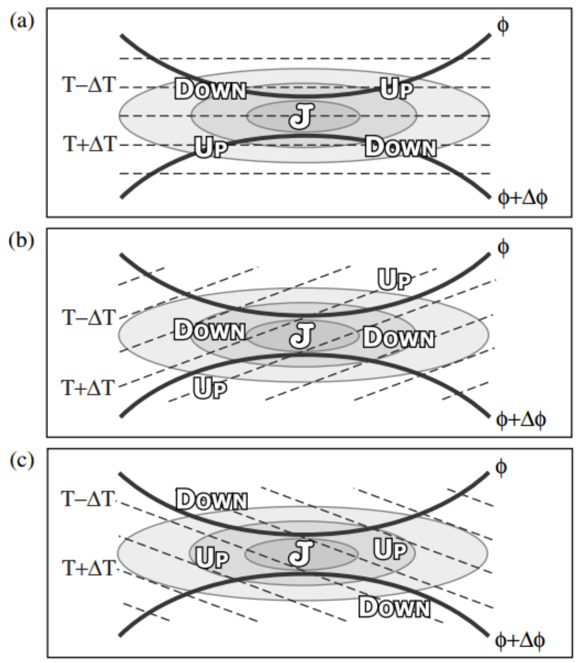

Figure 1. Schematic illustrations of an upper tropospheric jet associated with (a) no potential temperature advection, (b) cold potential temperature advection, and (c) warm potential temperature advection. In each panel, dashed lines represent isentropes while thick solid lines represent isohypses. Labels of "Up" and "Down" indicates regions of middle-to-upper tropospheric ascent and descent, respectively. Figure reproduced from Lang and Martin (2012, *Quart. J. Roy. Meteor. Soc.*), published by Wiley, their Figure 3.

As stated before, cold potential temperature advection on an isobaric surface implies sinking motion across that isobaric surface. Consequently, our first case is associated with quasi-geostrophic forcing for descent. This forcing is maximized in the jet core, where the wind component perpendicular to the isentropes or isotherms is maximized (and, thus, cold advection is maximized). Conversely, warm potential temperature advection on an isobaric surface implies rising motion across that isobaric surface. Consequently, our second case is associated with quasi-geostrophic forcing for ascent. This forcing is again maximized in the jet core, where the wind component perpendicular to the isentropes or isotherms is maximized (and, thus, warm advection is maximized).

Thus, there are two contributors to vertical motions we must consider: (1) parcel accelerations contributing to ageostrophic flow and vertical motion and (2) potential temperature advection by the geostrophic wind contributing to ageostrophic flow and vertical motion. In our first case, the combined effects of these two contributors result in descent aligned with the jet core; in the second case, the combined effects of these two contributors result in ascent aligned with the jet core. The interpretation of the four quadrant jet model is otherwise similar to that which we have considered before, as can be inferred from a comparison of Figure 1(b,c) to Figure 1(a).

Evaluation of the Quasi-Geostrophic Omega Equation

The quasi-geostrophic omega equation contains no partial derivatives with respect to time. As a result, this equation ***cannot*** be used to make a forecast! Instead, it may only be used to *diagnose* vertical motions (on the synoptic-scale under the constraints of the quasi-geostrophic system) at a given time.

Similar to the quasi-geostrophic height tendency equation, the vertical motion ω on the left-hand side of (5) depends upon the second derivative of ω with respect to x and y (as manifest through the Laplacian operator) as well as p. In other words, the local value of the vertical motion depends upon its value at adjacent locations in both the horizontal and vertical. Thus, to solve this system requires an iterative approach, one that can be difficult to code up and computationally expensive to execute. Also as before, it is difficult to accurately compute the frictional and diabatic heating forcing terms that make up part of the right-hand side of (5).

However, the differential geostrophic vorticity advection and Laplacian of potential temperature advection terms may be computed and/or estimated readily from any available source of atmospheric data, such as a numerical model analysis or forecast. This, in concert with the general proportionality stated in (6), enables us to diagnose synoptic-scale, mid-tropospheric vertical motions. Computational approaches are preferable over estimation approaches as the two primary forcing terms can – and often do! – oppose one another in sign, making their relative magnitudes important.