

Synoptic Meteorology II: Potential Vorticity Inversion and Anomaly Structure

14-16 April 2015

Readings: Sections 4.2 and 4.4 of *Midlatitude Synoptic Meteorology*.

Potential Vorticity Inversion

Introduction

One of the most important facets of potential vorticity is that if we know the three-dimensional distribution of potential vorticity, we can obtain information about all other fields. To some extent, we can see this solely by looking at the definition of the isentropic potential vorticity, restated below for convenience:

$$P = -g\eta \frac{\partial \theta}{\partial p} \quad (1)$$

The isentropic potential vorticity is a function of the absolute vorticity, which itself is dependent upon the u and v horizontal wind components, as well as the vertical change in potential temperature, which itself is related to the geopotential height field due to the relationship of potential temperature with the thickness of a layer. Therefore, it stands to follow that if we know the three-dimensional distribution of isentropic potential vorticity, we should be able to extract out information about u , v , θ , Φ , and so on. This is what is known as the *invertibility principle*.

To help illustrate this point, let us return to the definition of the geostrophic relative vorticity in terms of the geopotential height field, i.e.,

$$\zeta_g = \frac{1}{f_0} \nabla^2 \Phi \quad (2)$$

Throughout the semester, we have used this statement to *qualitatively* infer the distribution of geostrophic relative vorticity with respect to the geopotential height field (and vice versa). Under the constraint of geostrophic balance – inherent to the definition of both the geopotential height and the geostrophic relative vorticity – (2) may be “inverted” so as to pose the geopotential height in terms of the geostrophic relative vorticity, i.e.,

$$\Phi = f_0 \nabla^{-2} \zeta_g \quad (3)$$

Equation (3) states that if we know the distribution of the geostrophic relative vorticity field, we can *quantitatively* obtain the geopotential height field.

Principles of Potential Vorticity Inversion

The invertibility principle, as applied to more complex fields such as isentropic and Ertel potential vorticity, is a robust way for us to be able to assess the wind, geopotential height, and thermal fields associated with a given potential vorticity anomaly. While *qualitative* estimates for these fields can be obtained far more simply than through the use of the invertibility principle, it nevertheless has great utility in obtaining *quantitative* measures of these fields, particularly for specific meteorological features or disturbances.

It should be noted that neither the isentropic nor the Ertel potential vorticity may be inverted directly in their full form. In other words, we can't simply take the potential vorticity distribution and apply some operator to it to get out all of the other fields. Instead, we must first apply some sort of *balance relationship*, such as geostrophic balance, gradient wind balance, or even something more complex such as a non-linear balance. This concept of a balance condition can be viewed in the context of the relationship between the geostrophic relative vorticity and geopotential height. Inherent to that relationship is the application of geostrophic balance. Geostrophic balance enables us to define the geopotential height in terms of the wind by approximating the wind with the geostrophic wind.

More generally, a balance condition allows us to express the geopotential height and corresponding thermal fields in terms of the wind field (or vice versa). As applied to the isentropic potential vorticity as expressed by (1), this collapses its definition down to one unknown field, with the isentropic potential vorticity on the left-hand side and either the wind field or the geopotential height/thermal field on the right-hand side. At that point, the potential vorticity distribution may be inverted to obtain the field on the right-hand side, from which the other field may be obtained from the balance condition applied to the system.

In addition to a balance relationship, we also require *boundary conditions* to be able to invert the potential vorticity field. Why is this? Note the presence of partial derivatives with respect to pressure in the definitions of both the isentropic and Ertel potential vorticity. Thus, mathematically, in order to obtain the three-dimensional distribution of our fundamental variables (e.g., u , v , θ) from the three-dimensional distribution of potential vorticity, we must specify what those variables look like along the upper and lower boundaries of the three-dimensional volume.

Application of Potential Vorticity Inversion Principles

We can let gradient wind balance and hydrostatic balance provide balance conditions for the inversion problem. With some manipulation, the application of these balance conditions permits the development of a thermal wind-like relationship between the thermal and wind fields. This relationship permits the thermal fields – where a pressure gradient on an isentropic surface is equivalent to a temperature gradient on the isentropic surface, as we discussed before – to be expressed in terms of the wind field.

Operating upon the isentropic potential vorticity with this relationship enables an equation relating the horizontal wind speed V to the isentropic potential vorticity distribution to be obtained. That equation is simplified by making approximations that are consistent with the study of *relatively weak, synoptic-scale* isentropic potential vorticity anomalies. The resultant equation, given appropriate boundary conditions for V , can be solved to obtain the wind speed from the distribution of isentropic potential vorticity. Through use of the balance conditions noted above, the wind field can then be used to obtain the corresponding thermal fields, thereby completing the inversion process.

In the context of the invertibility principle, it can be stated that a given potential vorticity anomaly *induces* its accompanying wind field. Since the wind field is in balance with the thermal field as a result of the balance condition applied to the system, it can also be stated that a given potential vorticity anomaly induces its accompanying thermal structure. Later in this lecture, we will examine the vertical extent of and physical constraints upon these induced fields. Before doing so, however, we first examine the basic nature of the wind and thermal fields induced by weak, synoptic-scale upper-tropospheric potential vorticity and lower-tropospheric potential temperature anomalies.

The Structure of Upper Tropospheric Potential Vorticity Anomalies

To begin our consideration of the structure of upper tropospheric – or, truly, along-tropopause – potential vorticity anomalies, let us return to the basic definition of the isentropic potential vorticity presented in (1). In our last lecture, we defined *positive potential vorticity anomalies* as localized maxima in isentropic potential vorticity ($P > 0$). Likewise, we defined *negative potential vorticity anomalies* as localized minima in isentropic potential vorticity ($P < 0$).

By examining (1), we can describe the basic structure of upper tropospheric positive and negative potential vorticity anomalies:

- A **positive potential vorticity anomaly** is associated with locally large *cyclonic absolute vorticity* ($\eta > 0$) **and** *enhanced static stability* with tightly-packed isentropes in the vertical ($-\partial\theta/\partial p \gg 0$).
- A **negative potential vorticity anomaly** is associated with locally large *anticyclonic absolute vorticity* ($\eta < 0$) **and** *reduced static stability* with weakly-packed isentropes in the vertical ($-\partial\theta/\partial p > 0$).

With appropriate balance and boundary conditions, the specific distributions of absolute vorticity and static stability associated with a given potential vorticity anomaly may be obtained by inverting the potential vorticity, as described above. In what follows, we consider the structure of synoptic-scale potential vorticity anomalies obtained via potential vorticity inversion methods

described above. Hydrostatic and gradient wind balance are utilized as balance conditions, while the wind and thermal fields are constrained by appropriate boundary conditions.

We first examine the structure of a positive potential vorticity anomaly, as depicted in Figure 1. A positive potential vorticity anomaly is associated with a depressed height of the dynamic tropopause. Accompanying this is upper tropospheric cyclonic rotation. Isentropes are tightly packed in the vertical direction through the positive potential vorticity anomaly, indicative of high static stability. Thus, relatively warm potential temperature is found in the stratosphere above the positive anomaly and relatively cold potential temperature is found in the middle to upper troposphere below the positive anomaly. Likewise, above and below the positive potential vorticity anomaly, isentropes become less tightly packed in the vertical direction. This is indicative of weaker static stability at such altitudes.

Next, we examine the structure of a negative potential vorticity anomaly, as depicted in Figure 2. A negative potential vorticity anomaly is associated with a raised dynamic tropopause. Accompanying this is upper tropospheric anticyclonic rotation. Isentropes are weakly packed in the vertical direction through the negative potential vorticity anomaly, indicative of weak static stability. Thus, relatively cold potential temperature is found in the stratosphere above the negative anomaly and relatively warm potential temperature is found in the middle to upper troposphere below the negative anomaly. Likewise, above and below the negative potential vorticity anomaly, isentropes become more tightly packed in the vertical direction. This is indicative of stronger static stability at such altitudes.

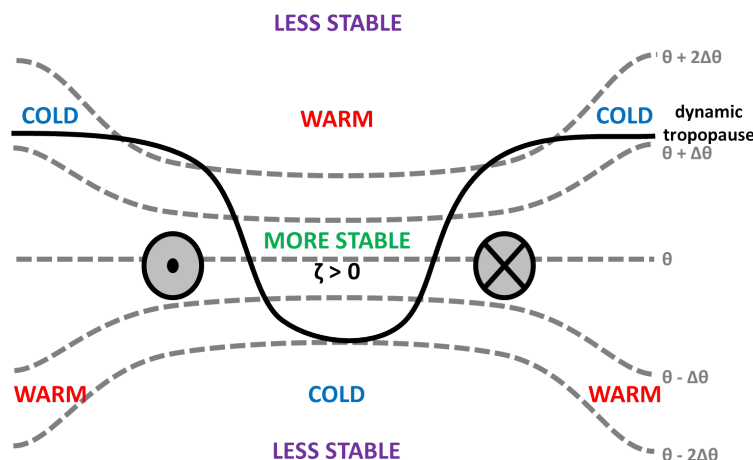


Figure 1. The structure of an idealized upper tropospheric positive potential vorticity anomaly. The dynamic tropopause is depicted by the solid black line while isentropes are depicted by the dashed grey lines. The circle with a dot in its center denotes flow out of the page while the circle with an X in its center denotes flow into the page.

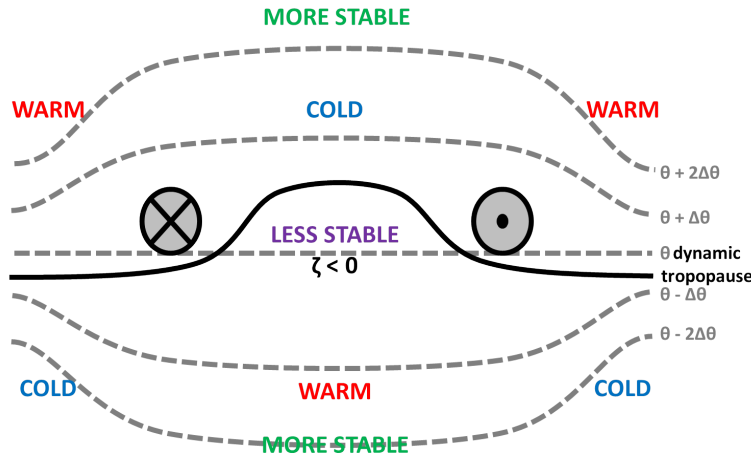


Figure 2. The structure of an idealized upper tropospheric negative potential vorticity anomaly. The dynamic tropopause is depicted by the solid black line while isentropes are depicted by the dashed grey lines. The circle with a dot in its center denotes flow out of the page while the circle with an X in its center denotes flow into the page.

The Structure of Surface Potential Temperature Anomalies

Analog to Upper Tropospheric Potential Temperature Anomalies

To find the full three-dimensional structure of the wind and thermal fields induced by a given potential vorticity anomaly, knowledge of the three-dimensional distribution of potential vorticity is required. This is not an issue away from the upper and lower boundaries of the inversion problem. However, since the wind and thermal fields are obtained from the potential vorticity field via usage of a partial differential equation, solutions for the wind and thermal fields must be specified along the upper and lower boundaries. These prescribed solutions are known as *boundary conditions*.

The upper boundary is typically taken to be somewhere in the stratosphere (e.g., $p = 60$ hPa), where synoptic-scale meteorological variability is negligible. The lower boundary is given by the surface, where we know that synoptic-scale meteorological variability is far from negligible. The boundary condition on this lower boundary is taken to be the surface potential temperature. Therefore, it stands to follow that surface potential temperature anomalies may act (i.e., have similar structure to) potential vorticity anomalies found along the dynamic tropopause.

As before, we find that with the specification of appropriate balance and boundary conditions, the specific distributions of absolute vorticity and static stability associated with a given surface potential temperature anomaly may be obtained by application of the invertibility principle. In obtaining the descriptions that follow, hydrostatic and gradient wind balance are again used as

balance conditions, while the wind and thermal fields are constrained by an appropriate upper boundary condition.

Surface Potential Temperature Anomaly Structure

Before we assess the structure of surface potential temperature anomalies, it is helpful to introduce the concept of “thermal vorticity.” In defining the thermal vorticity, we wish to recall the relationship between the geopotential height and geostrophic relative vorticity:

$$\zeta_g = \frac{1}{f_0} \nabla^2 \Phi \quad (4)$$

If we take the partial derivative of (4) with respect to p and commute the order of the partial derivatives on the right-hand side of the equation, we obtain:

$$\frac{\partial \zeta_g}{\partial p} = \frac{1}{f_0} \nabla^2 \frac{\partial \Phi}{\partial p} \quad (5)$$

Through substitution of the ideal gas law and Poisson’s equation, the hydrostatic equation may be written as:

$$\frac{\partial \Phi}{\partial p} = -h\theta \quad \text{where } h = \frac{R}{p_0} \left(\frac{p_0}{p} \right)^{\frac{c_v}{c_p}} \quad (6)$$

Substituting (6) into (5) and rearranging, we obtain:

$$f_0 \frac{\partial \zeta_g}{\partial p} = -h \nabla^2 \theta \quad (7)$$

Equation (7) is known as the *thermal vorticity relationship*. Under the constraints of quasi-geostrophic balance, the vertical derivative of the geostrophic relative vorticity ζ_g with respect to p is a function of the Laplacian of the potential temperature θ .

Since h is positive, given the known proportionality implied by the Laplacian on the right-hand side of (7), we know that $\partial \zeta_g / \partial p$ is a local maximum where θ is a local maximum. Similarly, $\partial \zeta_g / \partial p$ is a local minimum where θ is a local minimum. If we assume that ζ_g at some middle to upper tropospheric level away from the surface is zero, $\partial \zeta_g$ is negative for cyclonic geostrophic relative vorticity at the surface while $\partial \zeta_g$ is positive for anticyclonic geostrophic relative vorticity at the surface. Therefore, since $\partial p < 0$, θ is a local maximum where there is cyclonic geostrophic

relative vorticity at the surface and θ is a local minimum where there is anticyclonic geostrophic relative vorticity at the surface.

With this in mind, we first examine the structure of a positive (or warm) surface potential temperature anomaly, as depicted in Figure 3. Due to the thermal vorticity relationship expressed above, a warm surface potential temperature anomaly is associated with cyclonic rotation at the surface. Because potential temperature generally increases with height, a warm surface potential temperature anomaly implies a downward bowing of the near-surface isentropes. This leads to relatively strong static stability at and just above the surface and relatively weak static stability at higher altitudes. The combination of cyclonic rotation and relatively strong static stability at the surface thus suggests that warm surface potential temperature anomalies are conceptually equivalent to positive potential vorticity anomalies.

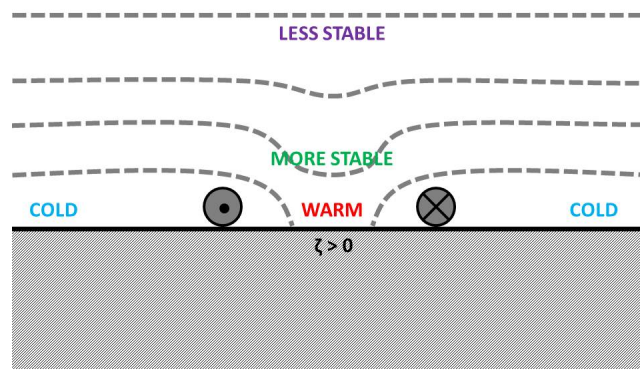


Figure 3. The structure of an idealized surface positive (warm) potential temperature anomaly. The surface is depicted by the solid black line while isentropes are depicted by the dashed grey lines. The circle with a dot in its center denotes flow out of the page while the circle with an X in its center denotes flow into the page.

Next, we examine the structure of a negative (or cold) surface potential temperature anomaly, as depicted in Figure 4. Due to the thermal vorticity relationship expressed above, a cold surface potential temperature anomaly is associated with anticyclonic rotation at the surface. Because potential temperature generally increases with height, a cold surface potential temperature anomaly implies an upward lifting of the near-surface isentropes. This leads to relatively weak static stability at and just above the surface and relatively strong static stability at higher altitudes. The combination of anticyclonic rotation and relatively weak static stability at the surface thus suggests that cold surface potential temperature anomalies are conceptually equivalent to negative potential vorticity anomalies.

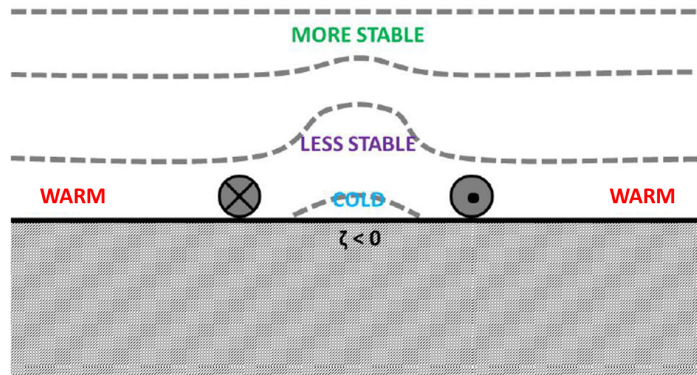


Figure 4. The structure of an idealized surface negative (cold) potential temperature anomaly. The surface is depicted by the solid black line while isentropes are depicted by the dashed grey lines. The circle with a dot in its center denotes flow out of the page while the circle with an X in its center denotes flow into the page.

The Vertical Sphere of Influence of Potential Vorticity Anomalies

When we described the structure of positive and negative potential vorticity anomalies above, we stated that they were weak, synoptic-scale anomalies. This gives us a measure of the horizontal length scale of such anomalies: on the order of 1,000 km or larger. However, it is worthwhile to also consider the vertical length scale of such anomalies. In other words, over how deep of a vertical layer are the wind and thermal fields associated with positive and negative potential vorticity anomalies communicated?

It can be shown that the vertical depth $\Delta\theta$ over which the wind and thermal fields are induced by a given potential vorticity anomaly can be expressed as:

$$\Delta\theta = \frac{f_0 L}{\sqrt{\sigma_{ref}^* g K_{ref}}} \quad (8)$$

In (8), L is the horizontal length scale, $f_0 = 1 \times 10^{-4} \text{ s}^{-1}$, $g = 9.81 \text{ m s}^{-2}$, σ_{ref}^* is an *inverse* measure of the reference-state static stability, and K_{ref} is related to the reference-state density.

The key take-home messages from (8) are:

- The vertical scale of the influence of a potential vorticity anomaly is directly proportional to its horizontal scale.

- Potential vorticity anomalies that are larger in horizontal extent induce wind and thermal fields over a greater vertical depth.
- Potential vorticity anomalies that are smaller in horizontal extent induce wind and thermal fields over a shallower vertical depth.
- The vertical scale of the influence of a potential vorticity anomaly is inversely proportional to σ_{ref}^* and, due to the definition of σ_{ref}^* as noted above, is directly proportional to the *reference-state* (e.g., environmental) static stability.
 - Potential vorticity anomalies embedded within environments of stronger static stability induce wind and thermal fields over a greater vertical depth.
 - Potential vorticity anomalies embedded within environments of weaker static stability induce wind and thermal fields over a shallower vertical depth.

If we plug in representative values for each of the terms of (8), we find that $\Delta\theta \approx 55\text{-}60$ K. This is typically as large as – if not larger than – the depth of the troposphere.

As a consequence, we state that a weak, synoptic-scale potential vorticity anomaly can impact the wind and thermal fields over a deep vertical depth. Feasibly, an upper tropospheric potential vorticity anomaly can influence the wind and thermal fields all the way down to the surface, while a surface potential temperature anomaly can influence the wind and thermal fields all the way up to the tropopause. This is known as the *action at a distance* principle. Naturally, the wind and thermal fields induced by a given potential vorticity (or potential temperature) anomaly are most coherent (i.e., strongest) at the altitude of the anomaly and gradually decay away from it in the vertical. We will return to this idea of “action at a distance” when we examine cyclone development from the potential vorticity perspective in a later lecture.

We can also repeat the above scaling exercise to get an estimate of the relative *strength* of the wind (and thermal) fields induced by a given potential vorticity anomaly. Doing so, the following relationship is obtained:

$$U = \sigma_{ref}^* PL \quad (9)$$

In (9), P represents the magnitude of the potential vorticity anomaly. The key take-home messages from (9) are:

- The strength of the wind and thermal fields induced by a given potential vorticity anomaly are directly proportional to the intensity of the potential vorticity anomaly.
 - A stronger positive or negative potential vorticity anomaly induces stronger wind and thermal fields.

- A weaker positive or negative potential vorticity anomaly induces weaker wind and thermal fields.
- The strength of the wind and thermal fields induced by a given potential vorticity anomaly are directly proportional to the horizontal scale of the potential vorticity anomaly.
 - Potential vorticity anomalies that are larger in horizontal – and, from the above, vertical – scale induce stronger wind and thermal fields.
 - Potential vorticity anomalies that are smaller in horizontal – and, from the above, vertical – scale induce weaker wind and thermal fields.
- The strength of the wind and thermal fields induced by a given potential vorticity anomaly are, due to the definition of σ_{ref}^* , inversely proportional to the reference-state (or environmental) static stability.
 - Weaker environmental static stability results in stronger induced wind and thermal fields.
 - Stronger environmental static stability results in weaker induced wind and thermal fields.
 - Thus, weak environmental static stability implies shallow yet stronger induced wind and thermal fields, while stronger environmental static stability implies deep yet weaker induced wind and thermal fields.

Finally, it should be noted that in the above discussion, we explicitly considered the vertical depth and impacts of upper tropospheric potential vorticity anomalies. However, the same insights hold for the surface potential temperature anomalies we described earlier as well. Thus, the vertical scale of a surface potential temperature anomaly is related to its horizontal scale and static stability, while the strength of its induced wind and thermal fields is related to its horizontal scale, the magnitude of the surface potential temperature anomaly, and the reference-state static stability. As with upper tropospheric potential vorticity anomalies, synoptic-scale surface potential temperature anomalies can induce wind and thermal fields over a great vertical depth spanning the tropopause (or nearly so).

Potential Vorticity Anomalies and the Correspondence Between Absolute Vorticity and Static Stability

As illustrated above for both upper tropospheric potential vorticity and surface potential temperature anomalies, absolute vorticity and static stability anomalies are of like “sense.” This means that cyclonic absolute vorticity is associated with enhanced static stability (and vice versa) and anticyclonic absolute vorticity is associated with reduced static stability (and vice versa). The fundamental reason for this is rooted in thermal wind balance.

To illustrate this correspondence between the absolute vorticity and static stability, let us consider a positive upper tropospheric potential vorticity anomaly. This anomaly is associated with cyclonic absolute vorticity maximized at the altitude of the potential vorticity anomaly. In other words, the rotational flow decreases in magnitude above and below the altitude of the potential vorticity anomaly.

If we take a vertical east-west cross-section through this anomaly, then the rotational flow is out of the north ($v < 0$) to the west of the anomaly and out of the south ($v > 0$) to the east of the anomaly. Above and below the altitude of the anomaly, the magnitude of v becomes smaller. To the west of the anomaly, this means that v is less negative above and below the altitude of the anomaly. Conversely, to the east, this means that v is less positive above and below the altitude of the anomaly.

The thermal wind is defined as the vector difference in the wind between two vertical levels. Thus, to the west of the anomaly, the thermal wind is positive above (smaller negative minus larger negative = positive) and negative below (larger negative minus smaller negative = negative) the altitude of the anomaly. Conversely, to the east of the anomaly, the thermal wind is negative above (smaller positive minus larger positive = negative) and positive below (larger positive minus smaller positive = positive) the altitude of the anomaly.

Recall that the meridional component of the thermal wind on an isobaric surface takes the form:

$$v_T = \frac{R}{f} \frac{\partial \bar{T}}{\partial x} \ln \left(\frac{p_0}{p_1} \right) \quad (10)$$

Since R , f , and $\ln(p_0/p_1)$ are all positive, the sign of the layer-mean zonal temperature gradient is the same as that on v_T . Thus, $\frac{\partial \bar{T}}{\partial x} > 0$ to the east below the altitude of the anomaly and to the west above the altitude of the anomaly. This is the situation in which warmer air is found along the positive x-axis, or to the east. Likewise, $\frac{\partial \bar{T}}{\partial x} < 0$ to the east above the altitude of the anomaly and to the west below the altitude of the anomaly. This is the situation in which warmer air is found along the negative x-axis, or to the west.

As a result,

- Above the altitude of the vortex: $\frac{\partial \bar{T}}{\partial x} > 0$ to the west, $\frac{\partial \bar{T}}{\partial x} < 0$ to the east
 - Warmer air is found immediately above the potential vorticity anomaly.
 - Colder air is found to the east and west above the potential vorticity anomaly.
- Below the altitude of the vortex: $\frac{\partial \bar{T}}{\partial x} > 0$ to the east, $\frac{\partial \bar{T}}{\partial x} < 0$ to the west
 - Colder air is found immediately below the potential vorticity anomaly.
 - Warmer air is found to the east and west below the potential vorticity anomaly.

Recall that temperature on an isobaric surface is directly analogous to potential temperature on that surface. Thus, while the above statements are posed in terms of temperature, they can equivalently be posed in terms of potential temperature.

Since potential temperature generally increases with increasing height, the presence of relatively warm air immediately above the potential vorticity anomaly implies that the isentropes bow downward there. Likewise, the presence of relatively cool air immediately below the potential vorticity anomaly implies that the isentropes bow upward there. This results in the static stability being relatively large in the heart of the positive potential vorticity anomaly – as expected!

This exercise can be repeated for an upper tropospheric negative potential vorticity anomaly; by working in reverse from a given static stability distribution to obtain the absolute vorticity distribution that must accompany it; and/or for surface potential temperature anomalies. These are exercises left to the interested student. Regardless, the take-home point is as stated above: upper tropospheric potential vorticity and surface potential temperature anomalies are associated with absolute vorticity and static stability structures of like sense.