

Synoptic Meteorology I: Thermal Wind Balance

9, 14 October 2014

Deriving the Thermal Wind Relationship

Recall from our most recent lecture that the geostrophic relationship applicable on isobaric surfaces can be expressed mathematically by:

$$fv = \frac{\partial\Phi}{\partial x} \quad (1a)$$

$$fu = -\frac{\partial\Phi}{\partial y} \quad (1b)$$

If we substitute v_g for v and u_g for u and divide both sides of (1) by f , we obtain:

$$v_g = \frac{1}{f} \frac{\partial\Phi}{\partial x} \quad (2a)$$

$$u_g = -\frac{1}{f} \frac{\partial\Phi}{\partial y} \quad (2b)$$

The *thermal wind* is defined as the vector difference in the geostrophic wind between two pressure levels p_1 and p_0 , where p_0 is closer to the surface (and thus $p_0 > p_1$). To obtain an expression for the thermal wind, we differentiate (2) with respect to pressure ($\partial/\partial p$) to obtain:

$$\frac{\partial v_g}{\partial p} = \frac{1}{f} \frac{\partial}{\partial p} \left[\frac{\partial\Phi}{\partial x} \right] \quad (3a)$$

$$\frac{\partial u_g}{\partial p} = -\frac{1}{f} \frac{\partial}{\partial p} \left[\frac{\partial\Phi}{\partial y} \right] \quad (3b)$$

If we commute the order of the partial derivatives on the right-hand sides of (3), then both equations contain a common $\partial\Phi/\partial p$ term. The hydrostatic equation states that:

$$\frac{\partial p}{\partial z} = -\rho g \rightarrow \frac{\partial p}{\rho} = -g \partial z$$

If we plug in to the hydrostatic equation with the definition of the geopotential Φ ($\partial\Phi = g \partial z$) and the ideal gas law ($\rho = p/RT$), we obtain the following expression:

$$\frac{\partial\Phi}{\partial p} = -\frac{RT}{p}$$

Substituting this into (3) above, we obtain:

$$\frac{\partial v_g}{\partial p} = \frac{1}{f} \frac{\partial}{\partial x} \left[-\frac{RT}{p} \right] \quad (4a)$$

$$\frac{\partial u_g}{\partial p} = -\frac{1}{f} \frac{\partial}{\partial y} \left[-\frac{RT}{p} \right] \quad (4b)$$

We can simplify (4) by taking R out of the derivatives as a constant. Furthermore, because we are using the form of the geostrophic wind applicable on isobaric surfaces, p is constant with respect to both x and y . We can thus extract p and put it into ∂p , where by definition, $\partial p/p = \partial(\ln p)$ and thus $p/\partial p = 1/\partial(\ln p)$. Doing so, we obtain:

$$\frac{\partial v_g}{\partial \ln p} = -\frac{R}{f} \frac{\partial T}{\partial x} \quad (5a)$$

$$\frac{\partial u_g}{\partial \ln p} = \frac{R}{f} \frac{\partial T}{\partial y} \quad (5b)$$

Equation (5) gives us expressions for the vertical wind shear of the geostrophic wind. If we multiply both sides of (5) by $\partial \ln p$ and integrate from p_0 to p_1 , we obtain:

$$\int_{p_0}^{p_1} \partial v_g = \int_{p_0}^{p_1} -\frac{R}{f} \frac{\partial T}{\partial x} \partial \ln p \quad (6a)$$

$$\int_{p_0}^{p_1} \partial u_g = \int_{p_0}^{p_1} \frac{R}{f} \frac{\partial T}{\partial y} \partial \ln p \quad (6b)$$

We note that R and f are both constant with respect to p and can thus be pulled out of the integrals. Likewise, if we substitute \bar{T} for T , where \bar{T} is a layer-mean (or vertically-averaged) temperature, then the derivative terms with respect to \bar{T} can be pulled out of the integrals. This is similar to what we did to obtain the hypsometric equation. Doing so, (6) becomes:

$$v_T \equiv v_g(p_1) - v_g(p_0) = -\frac{R}{f} \frac{\partial \bar{T}}{\partial x} (\ln(p_1) - \ln(p_0)) = \frac{R}{f} \frac{\partial \bar{T}}{\partial x} \ln\left(\frac{p_0}{p_1}\right) \quad (7a)$$

$$u_T \equiv u_g(p_1) - u_g(p_0) = \frac{R}{f} \frac{\partial \bar{T}}{\partial y} (\ln(p_1) - \ln(p_0)) = -\frac{R}{f} \frac{\partial \bar{T}}{\partial y} \ln\left(\frac{p_0}{p_1}\right) \quad (7b)$$

Equation (7) defines the *thermal wind*. It states that the change in geostrophic wind between two isobaric levels p_1 and p_0 is directly related to the horizontal layer-mean temperature gradient.

Because the layer-mean temperature is directly proportional to the thickness, we can equivalently state that the thermal wind is directly related to the horizontal thickness gradient.

Inspecting (7), we see that the meridional component of the thermal wind is a function of the zonal gradient in the layer-mean temperature. Likewise, the zonal component of the thermal wind is a function of the meridional gradient in the layer-mean temperature. Compare this to (3) and (11): the meridional component of the geostrophic wind is a function of the zonal gradient of either pressure or height, while the zonal component of the geostrophic wind is a function of the meridional gradient of either pressure or height.

We stated then that the geostrophic wind blows parallel to the isobars or lines of constant height. Likewise, we can now state that *the thermal wind “blows” parallel to the isotherms* (of layer-mean temperature). Equivalently, we can state that the thermal wind “blows” parallel to lines of constant thickness. This means that if we know the thermal wind, we know how the isotherms and lines of constant thickness are oriented spatially.

In the Northern Hemisphere, warm air is found 90° to the *right* of an observer facing downstream of the thermal wind. We can prove this to ourselves using (7). Consider a thermal wind “blowing” from south to north ($v_T > 0$). On the right-hand side of (7a), R , f , and $\ln(p_0/p_1)$ are all positive. As a result, $\frac{\partial \bar{T}}{\partial x}$ must also be positive, meaning that \bar{T} is warmer along the positive x -axis (i.e., to the east). This is 90° to the right of an observer facing downstream (here, to the north) of the thermal wind. Similar arguments can be made for any given v_T .

Application to Horizontal Temperature Advection

If we know the geostrophic wind at two pressure levels, we can determine the thermal wind over the vertical layer between those two levels using vector subtraction. Given the thermal wind, we know how the isotherms are oriented horizontally, as described above. From this information, we can analyze how the geostrophic wind at each pressure level blows with respect to the isotherms to qualitatively assess the mean horizontal temperature advection within the vertical layer. These principles are illustrated graphically below in Figure 1.

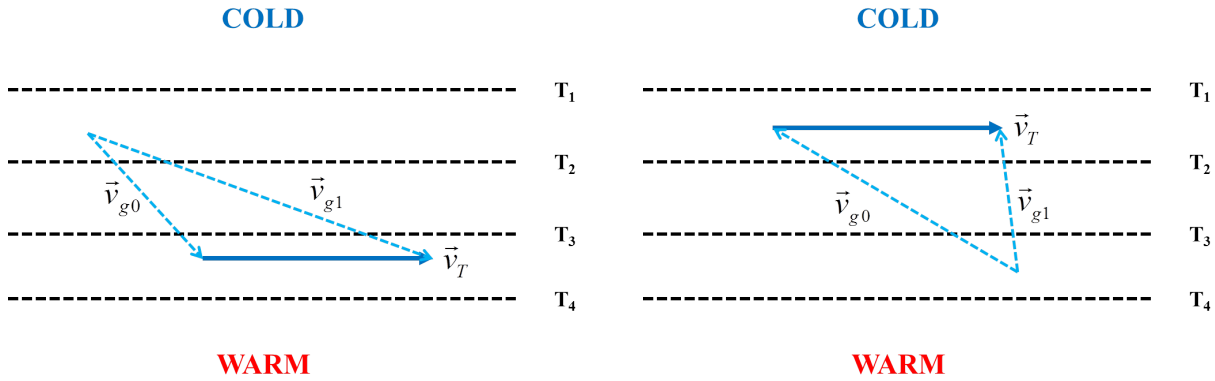


Figure 1. (left) A thermal wind v_T that is associated with cold air advection. (right) A thermal wind v_T that is associated with warm air advection. Note how the thermal wind is of identical direction and magnitude between the two panels but that the sign of the horizontal temperature advection is different. This is due to how the geostrophic wind within the layer defined by the thermal wind is oriented with respect to the isotherms of layer-mean temperature.

In the left-most panel of Figure 1, the geostrophic wind \vec{v}_{g0} at p_0 (the lower level) is from the northwest. The geostrophic wind \vec{v}_{g1} at p_1 (the upper level) is from the west-northwest. The geostrophic wind is turning counterclockwise with increasing distance above the ground, which we refer to as *backing* winds.

At this point, it is helpful to recall basic tenets of vector addition and subtraction. Two vectors \vec{A} and \vec{B} can be added together by placing the beginning of \vec{B} at the end of \vec{A} and drawing a vector from the beginning of \vec{A} to the end of \vec{B} . The difference $\vec{A} - \vec{B}$ may be viewed as $\vec{A} + (-\vec{B})$, where $(-\vec{B})$ is simply \vec{B} rotated by 180°. Equivalently, $\vec{A} - \vec{B}$ may be obtained by placing the beginning of \vec{B} at the beginning of \vec{A} and drawing a vector from the end of \vec{B} to the end of \vec{A} .

We now return to our examples in Figure 1. The vector difference $\vec{v}_{g1} - \vec{v}_{g0}$, defining the thermal wind, is obtained by placing the two vectors at a common origin, then drawing a vector from the end of \vec{v}_{g0} to the end of \vec{v}_{g1} . By definition, the (layer-mean) isotherms are parallel to this vector with the warm air found to the right (here, the south). Because the geostrophic wind at both p_0 and p_1 blows from cold toward warm air, backing of the geostrophic wind with increasing distance above the ground is associated with cold air advection.

Similarly, in the right-most panel of Figure 1, the geostrophic wind \vec{v}_{g0} at p_0 is from the southeast. The geostrophic wind \vec{v}_{g1} at p_1 is from the south-southeast. The geostrophic wind is turning clockwise with increasing distance above the ground, which we refer to as *veering* winds.

The vector difference between the two is again obtained by placing the two vectors at a common origin and drawing a vector from the end of \vec{v}_{g0} to the end of \vec{v}_{g1} . By definition, the isotherms are parallel to this vector with the warm air found to the right (again, the south). Because the geostrophic wind at both p_0 and p_1 blows from warm toward cold air, veering of the geostrophic wind with increasing distance above the ground is associated with warm air advection.

A great way to put these principles into practice is to regularly look at skew- $T/\ln p$ charts from one or more location(s). If you approximate the geostrophic wind by the total wind, how it changes with height will give you an approximate idea of whether cold or warm air advection is ongoing within a given layer at a given location without needing any other data. This principle is demonstrated in the accompanying “Thermal Wind Application” notes.

Further Insights from the Thermal Wind

Under the constraint of thermal wind balance, as manifest by (7), the following tenets are true:

- Where there is a change in geostrophic wind (speed and/or direction) with height, there must be a horizontal gradient in layer-mean temperature.
- Where there is a horizontal gradient in layer-mean temperature, there must be a change in geostrophic wind (speed and/or direction) with height.

The term *vertical wind shear* is commonly used to refer to changes in wind speed and/or direction with height. Recall from our discussion of (5) above that the thermal wind can be viewed as a kind of vertical wind shear, here of the geostrophic wind. Thus, thermal wind balance can be viewed as the balance between vertical wind shear and horizontal gradients of layer-mean temperature or thickness. In general, if one were to analyze synoptic-scale observations, one would find that this balance holds more often than not in the mid-latitudes.

We can also apply concepts thermal wind-related concepts to understanding Earth’s general circulation, particularly within the mid-latitudes. Consider that, in an annual average, latitudes closer to the Equator receive more insolation than do latitudes closer to the poles. As a result, again in an annual average, it is warmer throughout the troposphere closer to the Equator – say, at 30°N – than it is closer to the poles – say, at 60°N. This means that there exists a horizontal layer-mean temperature gradient. From the hypsometric equation, it also means that the thickness between two isobaric surfaces located near the surface (e.g., 1000 hPa) and the tropopause (e.g., 200 hPa) is greater near the Equator than near the poles. This is graphically depicted in Figure 2.

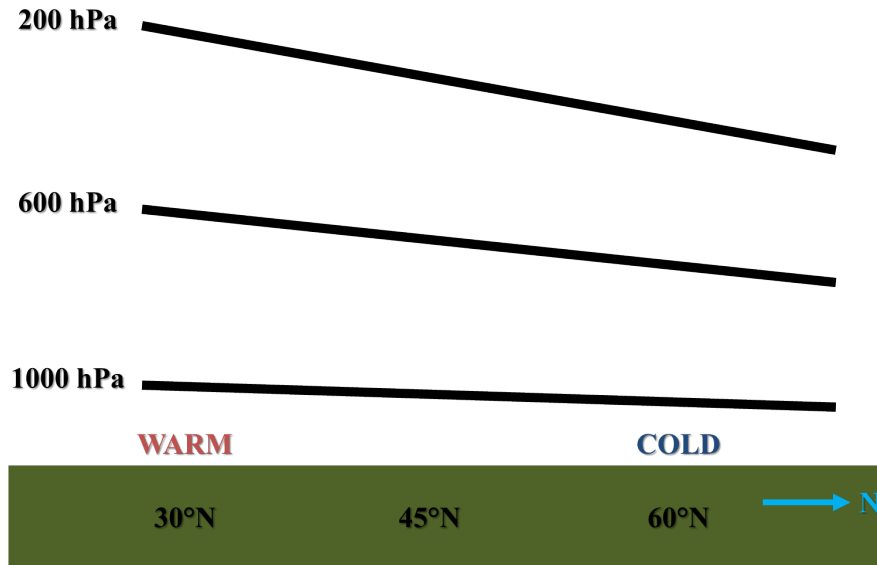


Figure 2. Annually-averaged vertical cross-section between 30°N and 60°N of three selected isobaric surfaces (200, 600, and 1000 hPa; black lines) and the relative layer-mean temperature between the 1000 hPa and 200 hPa isobaric surfaces at 30°N (**WARM**) and 60°N (**COLD**).

The north-south gradient in thickness – i.e., layer-mean temperature – means that the thermal wind for this layer is directed from west to east, such that warm air is found 90° to the right of an observer with the thermal wind at their back. This defines the vector difference in the geostrophic wind within the layer being considered. Note that a non-zero thermal wind defines what is known as a *baroclinic* atmosphere.

We can go a step further, however. Recall that geostrophic balance is defined as the balance between the horizontal pressure gradient force and the Coriolis force. In Figure 2, the horizontal (here, north-south) gradient in pressure is greater at higher altitudes than it is closer to the surface. This implies that the horizontal pressure gradient force is greater at higher altitudes, thus requiring a greater Coriolis force to maintain geostrophic balance. Because the magnitude of the Coriolis force is directly proportional to the horizontal wind speed, this implies a faster horizontal wind speed at higher altitudes than closer to the surface.

Furthermore, the horizontal pressure gradient is directed from north to south, such that the horizontal pressure gradient force is directed from south to north. The Coriolis force, under the constraint of geostrophic balance, is thus directed from north to south. Because we know that the Coriolis force is directed 90° to the right of the wind, we know that the geostrophic wind is directed from west to east at all altitudes. Because the horizontal wind speed – or magnitude of the geostrophic wind vector – is larger at higher altitudes, this implies westerly geostrophic winds that increase with increasing distance above the ground. This is a defining characteristic of the mid-latitudes and is frequently observed on the synoptic- to planetary-scales.

For Further Reading

Sections 1.4.1 and 1.4.2 of *Midlatitude Synoptic Meteorology* by G. Lackmann derives the thermal wind relationship and relates the thermal wind to the mean temperature advection within a given vertical layer. Section 4.3 of *Mid-Latitude Atmospheric Dynamics* by J. Martin provides a derivation and discussion of thermal wind balance and provided much of the source material for the applications of thermal wind balance discussed herein.