<u>Synoptic Meteorology I: Hydrostatic Balance, the Hypsometric Equation, and Thickness</u>

For Further Reading

Section 1.4 of *Midlatitude Synoptic Meteorology* by G. Lackmann derives the hypsometric equation and introduces thickness and its applications. Section 3.1 of *Mid-Latitude Atmospheric Dynamics* by J. Martin provides a basic derivation of the hydrostatic equation and a full derivation of the hypsometric equation. Section 6-1 of *Weather Analysis* by D. Djurić provides an in-depth discussion of how thickness may be used to identify fronts. Most other dynamic meteorology texts also include derivations and discussions of the hydrostatic and hypsometric equations.

Derivation of the Hydrostatic Equation

Consider a unit volume (V = 1 m^3, with sides of 1 m each) of air in the troposphere that is <u>*at rest*</u>. Assume that the horizontal properties of the air within the volume are uniform. In the absence of friction and the vertical component of the Coriolis force, there are two forces acting in the vertical on this unit volume of air: one related to pressure and one related to the weight of the air volume (or, more specifically, to gravity).

Recall that:

$$F = pA \tag{1}$$

$$W = gm = g\rho V \tag{2}$$

In (1) and (2), F = force (N), p = pressure (Pa), A = area (m^2), W = weight (kg m s^{-2} = N), g = 9.81 m s^{-2} (the gravitational constant), V = volume (m^3), and ρ = density (kg m^{-3}). The unit N refers to a Newton, equivalent to 1 kg m s^{-2}. Consequently, weight W is equivalent to a force.

We now wish to consider the forces acting on our air volume, as in Figure 1 below. There are two forces acting in the *downward* direction: weight, or the force associated with gravity ($g\rho A\Delta z$, noting that $V = A\Delta z$ by definition), and the pressure force acting upon the top of the air volume ($p(z+\Delta z)A$). There is a single force acting in the *upward* direction: the pressure force acting upon the bottom of the air volume ($p(z)A$). For both pressure forces, the quantities in parentheses denote the altitudes at which the pressure is valid, not multiplication operators.

In the atmosphere, for an atmosphere at rest (but also frequently also when the atmosphere is not at rest) the upward and downward forces are said to ***balance*** or cancel each other out. We can write this mathematically as:

$$p(z)A - p(z+\Delta z)A - g\rho A\Delta z = 0 \tag{3}$$

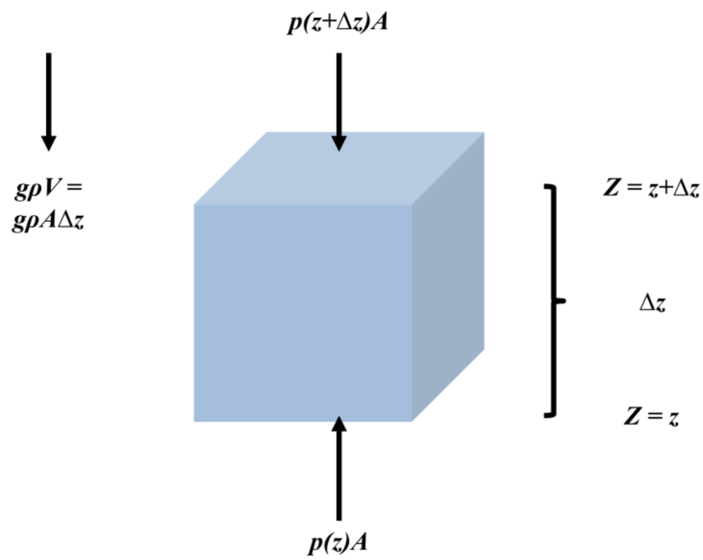

Figure 1. Graphical depiction of the forces acting in the vertical direction upon a unit volume of air. In the above, *p(z)* refers to the pressure at an altitude $Z = z$ while *p(z+Δz)* refers to the pressure at an altitude $Z = z + \Delta z$. In this example, volume V is written as the product of the area A and the height of the air volume Δz.

Note that the upward-directed force is prefaced with a positive sign and that the downward-directed forces are prefaced with negative signs. The idea of balance means that the addition of these forces must be equal to zero, such that the right-hand side of (3) is simply 0.

Next, divide (3) by $A\Delta z$ and group the pressure force terms to obtain:

$$\frac{p(z) - p(z + \Delta z)}{\Delta z} = \rho g \tag{4}$$

Multiplying by -1, we obtain:

$$\frac{p(z + \Delta z) - p(z)}{\Delta z} = -\rho g \tag{5}$$

If we take the limit of (5) as Δz approaches 0, we obtain:

$$\lim_{\Delta z \to 0} \frac{p(z + \Delta z) - p(z)}{\Delta z} = -\rho g \tag{6}$$

The left-hand side of (6) is equivalent to the partial derivative of p with respect to z, such that:

$$\frac{\partial p}{\partial z} = -\rho g \tag{7}$$

Equation (7) is the ***hydrostatic equation***. It provides a formulaic representation of what is known as ***hydrostatic balance***, describing the balance between the downward-directed gravitational force and the upward-directed pressure gradient force. Recall that the pressure gradient force is always directed from higher pressure toward lower pressure. Since pressure is a function of the mass of air that is above you, pressure is highest at ground level and decreases upward from there. Thus, the vertical component of the pressure gradient force is *always* directed upward.

Newton's Second Law of Motion states that the net force that is imposed on an object is equal to its mass times its acceleration. Stated differently, an object's acceleration is equal to the net force imposed upon the object divided by the object's mass. Under the constraint of hydrostatic balance, where the net force in the vertical direction is zero, air does not accelerate upward or downward. Explicitly, hydrostatic balance only holds when the vertical motion is zero; however, in practice, we find that hydrostatic balance holds when vertical motion is small or weak. On the synoptic-scale, this often is the case. Within thunderstorms, this is not necessarily true; this is beyond the scope of this class, however.

An alternative derivation of the hydrostatic equation can be obtained by performing a scale analysis of the vertical momentum equation for synoptic-scale motions. This derivation has the advantage of quantitively justifying neglecting friction and the vertical component of the Coriolis force in the derivation; however, it does not start from basic physical principles as does the derivation here. As a result, the alternative derivation is left as an exercise for the interested student.

Derivation of the Hypsometric Equation

Recall that the ideal gas law applicable when the air contains a non-zero amount of water vapor can be expressed as:

$$p = \rho R_d T_v \tag{8}$$

In (8), p = pressure (Pa), ρ = density (kg m^{-3}), Rd = dry air gas constant (287.04 J kg^{-1} K^{-1}), and T_v = virtual temperature (K). The virtual temperature can be approximated by $T_v = T(1 + 0.61w)$, where w = mixing ratio of water vapor (kg kg^{-1}). For common values of w of less than 0.02 kg kg^{-1} within the lower troposphere, T_v is equal to or slightly larger than T.

If we solve (8) for ρ and substitute into the hydrostatic equation (7), we obtain:

$$\frac{\partial p}{\partial z} = -\frac{pg}{R_d T_v} \tag{9}$$

Multiplying both sides of (9) by ∂z and dividing both sides of (9) by p, we obtain:

$$\frac{\partial p}{p} = -\frac{g \partial z}{R_d T_v} \tag{10}$$

If we solve (10) for ∂z and make the substitution of $\partial(\ln p)$ for $\partial p/p$, we obtain:

$$-\frac{R_d T_v}{g} \partial(\ln p) = \partial z \tag{11}$$

If we integrate (11) between pressure levels p_1 and p_2, where $p_1 > p_2$, at which the heights are z_1 and z_2, where $z_1 < z_2$, we obtain:

$$\int_{p_2}^{p_1} \frac{R_d T_v}{g} \partial(\ln p) = \int_{z_1}^{z_2} \partial z \tag{12}$$

Note that we have changed the order of the integration on the left-hand side (p_2 to p_1), which permits us to drop the leading negative sign. On the left-hand side of (12), we have two constants with respect to p: R_d and g. However, T_v is not constant with respect to p – in fact, it is far from constant with respect to p! This poses a problem, one that we get around by approximating T_v by a layer-mean value $\overline{T_v}$ that is constant with respect to p. If we do so and integrate both sides of (12), we obtain:

$$\frac{R_d \overline{T_v}}{g} \left(\ln(p_1) - \ln(p_2)\right) = z_2 - z_1 \tag{13}$$

If we combine the natural logarithms in (13) into a single term, we obtain:

$$\frac{R_d \overline{T_v}}{g} \ln\left(\frac{p_1}{p_2}\right) = z_2 - z_1 \tag{14}$$

Equation (14) is the ***hypsometric equation***. Because $p_1 > p_2$, the natural logarithm on the left-hand side of (14) is positive-definite (i.e., is always positive). The constants R_d and g are also positive-definite. This enables us to simply (14) to the following proportionality:

$$\overline{T_v} \propto z_2 - z_1 \tag{15}$$

This means that the difference in height $z_2 - z_1$ between two pressure surfaces p_1 and p_2, which we refer to as ***thickness*** ($z_2 - z_1 = \Delta z$), is directly proportional to the mean virtual temperature between the two pressure surfaces p_1 and p_2. This is a powerful statement, one that has many applications to understanding the Earth's atmosphere as well as synoptic-scale meteorological phenomena! We next consider several applications of this relationship.

Meteorological Applications of the Hypsometric Equation

The Height of Tropospheric Isobaric Surfaces

If we take $z_1 = z_{surface} = 0$ m, such that $p_1 = p_{surface}$, then (15) tells us that the height z_2 of some isobaric surface p_2 within the troposphere is higher when the layer-mean (virtual) temperature is warmer. Conversely, the height z_2 of the isobaric surface p_2 is lower when the layer-mean (virtual) temperature is colder. Consider, for example, the 500 hPa isobaric surface. The height of the 500 hPa isobaric surface is higher where the layer-mean temperature is warmer and lower where the layer-mean temperature is colder. When the layer-mean temperature rapidly changes in the zonal and/or meridional directions, so too does the height of the 500 hPa isobaric surface.

Let's apply this on the planetary-scale. Due to the incoming solar radiation imbalance (in an annually-averaged sense) between the poles and Equator, air temperature at and above the surface is typically coldest at the poles and increases as you move toward the Equator. Thus, on the planetary-scale, we would expect the height of the 500 hPa isobaric surface to be highest at the Equator and lowest at the poles. Thus, when one analyzes height on the 500 hPa isobaric surface (or, more generally, any tropospheric isobaric surface), the lowest heights are generally found at higher latitudes and the highest heights are generally found at lower latitudes.

Precipitation Type Analysis and Forecasting

Rawinsonde observations provide measurements of the height of isobaric surfaces above the ground. These data can be used to analyze the thickness between any two isobaric surfaces, such as 1000 hPa and 500 hPa or 1000 hPa and 850 hPa. Data obtained from numerical weather prediction model forecasts can be used to do the same for forecast data. Because the thickness between two isobaric surfaces is directly proportional to the mean (virtual) temperature within the vertical layer between the two isobaric surfaces, precipitation type may be crudely diagnosed from analyses and forecasts of thickness.

A commonly-used rule of thumb states that when the thickness of the 1000 hPa to 500 hPa layer is less than 5400 m, snow (rather than rain) is the most likely precipitation type. We can use (14) to "prove" this rule of thumb. Plugging in 1000 hPa for p_1, 500 hPa for p_2, 5400 m for $z_2 - z_1$, and the known values for R_d and g, we obtain an approximate value for $\overline{T_v}$ of 266.25 K (-6.9°C). Because $T \le T_v$, the approximate layer-mean temperature within this 1000 hPa to 500 hPa layer is less than or equal to -6.9°C. We thus might reasonably expect that it is cold enough within this layer to support snow reaching the surface (assuming that precipitation is possible or occurring).

Apart from the 1000 hPa to 500 hPa layer, the 1000 hPa to 850 hPa and 850 hPa to 700 hPa layers are also commonly used to diagnose precipitation type. The application of these two layers for the diagnosis of precipitation type, as well as both horizontal temperature advection and atmospheric stability, is described in the "Thickness and Precipitation Type" document available on the course website. Note, however, that precipitation type is crucially dependent upon the vertical temperature profile between cloud base and the ground, such that a full diagnosis of precipitation type requires analysis of observed or forecast skew *T*-ln *p* diagrams.

Application to Frontal Analysis

While we will discuss fronts and frontal analysis in great detail later this semester, it is useful to introduce a few basic aspects of frontal structure now given how readily they can be identified from thickness data. A front, or frontal zone, represents the dividing zone between two distinct air masses. Typically, fronts separate a warm and often moist air mass from a cold and often dry air mass. The two most commonly-observed types of fronts are cold fronts and warm fronts. Cold fronts are located at the leading edge of an advancing cold air mass, while warm fronts are located at the back edge of a retreating cold air mass.

Cold fronts slope upward in the rearward direction; for example, a cold front moving to the southeast is found at progressively higher altitudes the further northwest that you move away from the surface cold front. Warm fronts, by contrast, slope upward in the forward direction; for example, a warm front moving to the north is found at progressively higher altitudes the further north that you move away from the surface warm front. Thus, both types of fronts slope over the coldest air at the surface. Cold fronts slope more steeply – roughly 1 km up over 100 km horizontal distance – than do warm fronts – roughly 1 km up over 200 km horizontal distance.

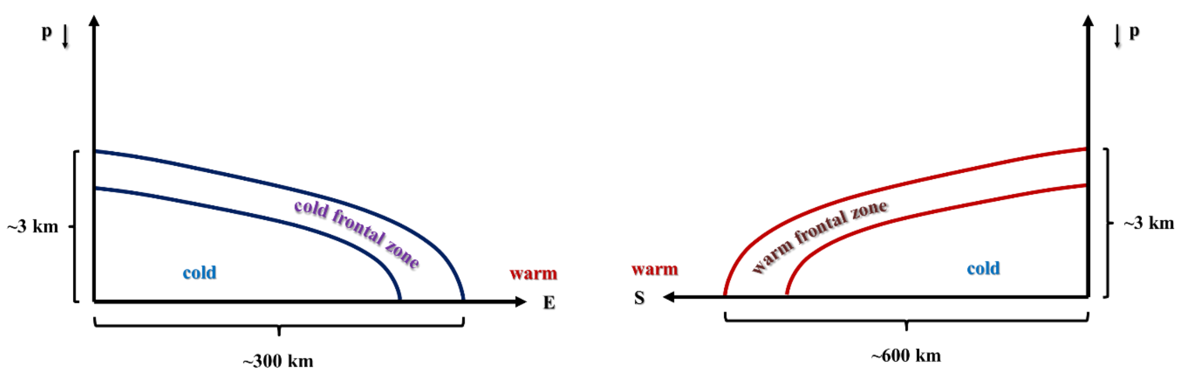

Figure 2. (left) Idealized vertical cross-section through a cold frontal zone. (right) Idealized vertical cross-section through a warm frontal zone. The directions of the lower axis are somewhat arbitrary, albeit chosen to reflect commonly-observed structures within the Northern Hemisphere. Note the steeper vertical slope of the cold frontal zone, as indicated by the axis labels. In both

examples, temperature decreases within increasing height at all locations along the lower axis except within the frontal zone, where a temperature inversion is typically present.

Let us now consider how thickness varies from east to west across the cold frontal zone depicted in Figure 2 above. Ahead of the cold front, within the warm air mass, we expect thickness to be relatively high. As we enter the cold frontal zone, the temperature near the surface cools compared to that ahead of the cold front but does not do so above the frontal zone. Thus, we expect thickness to be lower here, but not substantially so. As we move progressively further west, however, the coldness of the air – and the vertical depth over which it is found – increase. Thus, we expect thickness to continue to decrease. Similar arguments can be made for the warm frontal zone depicted in Figure 2 above as well.

Suppose we have a spatial analysis of thickness, perhaps of the 1000 hPa to 500, 700, or 850 hPa layer, upon which we've drawn lines of constant thickness. We would expect the smallest values of these isolines to be found well behind a cold front and/or well ahead of a warm front and the largest values of these isolines to be found ahead of a cold front and/or behind a warm front. We would further expect that the biggest changes in the values of these isolines over some horizontal distance would be found along and behind a cold and/or warm front. Applying these principles to our spatial analysis, we can obtain a reasonable guess for the location(s) of the cold and/or warm front(s) at the surface. An example of doing so may be found in Figure 3 below.

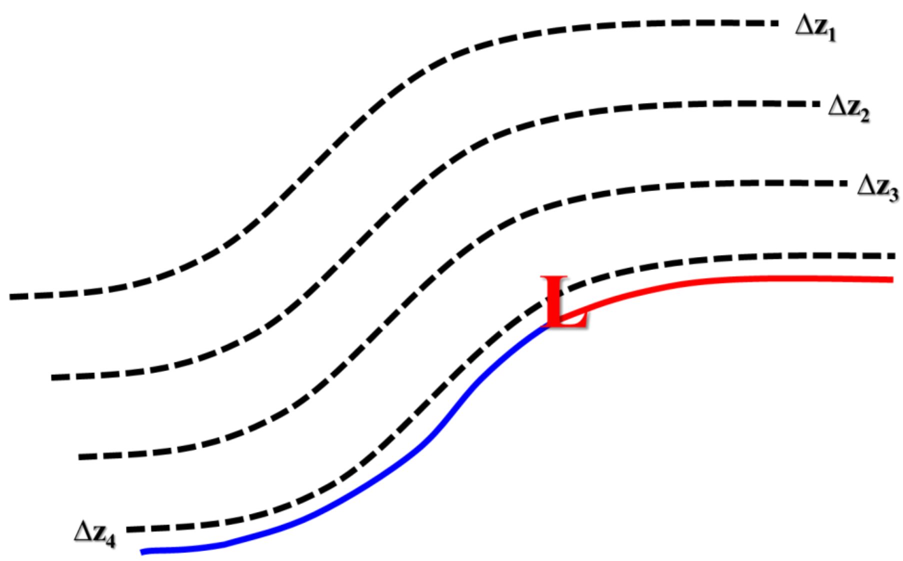

Figure 3. Idealized representation of 1000 hPa to 500 hPa layer thickness contours (dashed lines, with $\Delta z_1 < \Delta z_2 < \Delta z_3 < \Delta z_4$) and a surface cold (blue) and warm (red) front. In practice, slight deviations in frontal placement from the leading edge of the thickness contours are possible.

The Vertical Structure of Cyclones and Anticyclones

Thickness is a powerful tool by which the vertical structure of both cyclones and anticyclones may be understood. Let us consider two examples…

a) "An area of low pressure at the surface found within a warm air column disappears quickly with height."

Earlier, we stated that the thickness of the layer between two isobaric surfaces is directly proportional to the mean virtual temperature within that layer. Here, we have a warm air column, and thus we would expect the thickness within this column to be large compared to locations outside of this column. This means that, within the warm air column, isobaric surfaces at the bottom of the layer will be depressed downward toward the ground compared to locations outside of the warm air column. Isobaric surfaces at the top of the layer within the warm air column will be elevated upward compared to locations outside of the warm air column.

Consequently, at and near the surface, the pressure within the warm air column will be lower than outside of the warm air column. However, as you move upward, the pressure within the warm air column becomes larger than outside of the warm air column. See also Figure 4 below. Real-world examples of areas of low pressure at the surface found within columns of warm air include tropical cyclones and heat lows. The relative warmth found at the core of these features, coupled with their reduced intensity with increasing height, give rise to the term *warm-core cyclones* to describe these features.

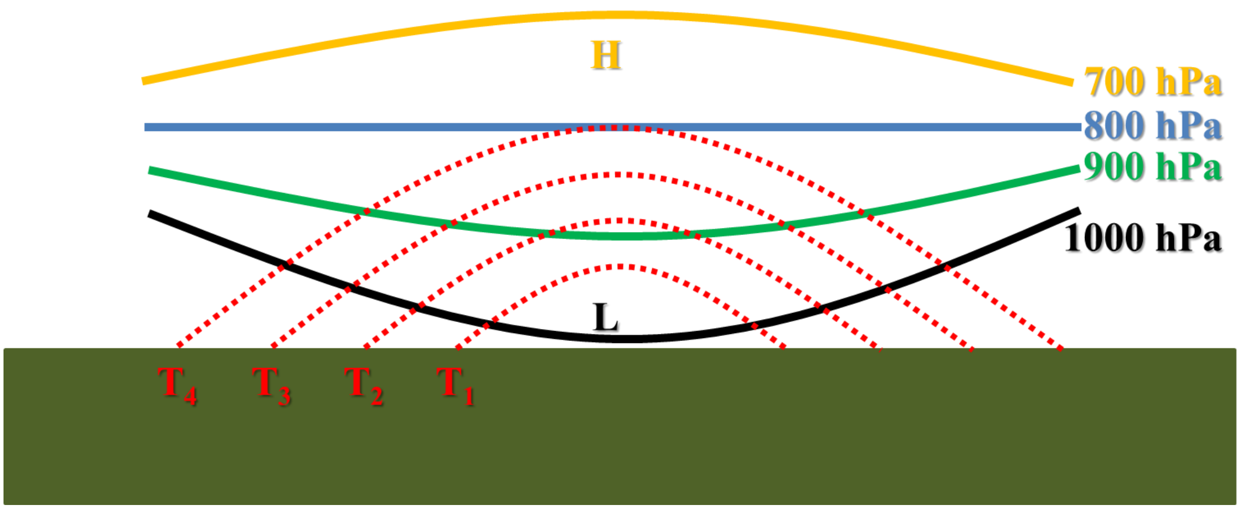

Figure 4. Schematic meant to accompany example (a) above. The red lines denote representative isotherms, where $T_1 > T_2 > T_3 > T_4$, such that the warmest temperatures are found in the center of the figure. Four isobaric surfaces, 700 hPa (orange), 800 hPa (blue), 900 hPa (green), and 1000 hPa (black), are given by solid lines. Note the greater vertical spacing between the isobaric surfaces within the warm air column as compared to outside the warm air column. Thus, an area of locally lower pressure at and near the surface weakens and disappears with increasing altitude.

b) "An area of low pressure at the surface found within a cold air column increases in intensity with height."

This is the opposite of what we described in example (a) above: here, we have an area of low pressure at the surface, but now it is found within a cold air column as opposed to a warm air column. We would thus expect the thickness of a layer between two isobaric surfaces within this column to be lower than between the same two isobaric surfaces outside of this column. Consequently, the intensity of the area of low pressure becomes stronger. See also Figure 5.

Real-world examples of areas of low pressure at the surface found within cold air columns include mid-latitude cyclones, or those associated with fronts, that we will study extensively this semester and next. The relatively coolness found at the heart of these features, coupled with their increasing intensity with increasing height, give rise to the term *cold-core cyclones* to describe these features. Further examples are given to you to interpret as a homework assignment.

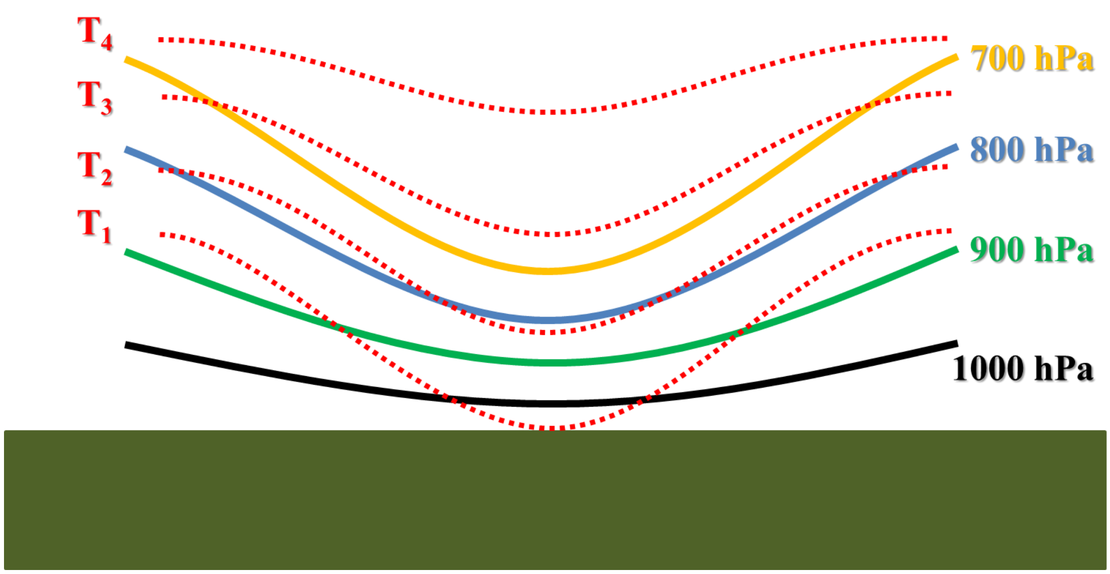

Figure 5. Schematic meant to accompany example (b) above. The red lines denote representative isotherms, where $T_1 > T_2 > T_3 > T_4$, such that the coldest temperatures are found in the center of the figure. Four isobaric surfaces, 700 hPa (orange), 800 hPa (blue), 900 hPa (green), and 1000 hPa (black), are given by solid lines. Note the smaller vertical spacing between the isobaric surfaces within the cold air column as compared to outside the cold air column. Thus, an area of locally lower pressure at and near the surface becomes more intense with increasing altitude.