Synoptic Meteorology I: The Geostrophic Approximation

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The Equations of Motion

In their most general form, and presented without formal derivation, the equations of motion applicable on constant height surfaces and posed in Cartesian coordinates are given by:

$$\frac{D\vec{v}}{Dt} = -2\vec{\Omega} \times \vec{v} - \frac{1}{\rho}\nabla p + \vec{g} + \vec{F}_r \quad (1)$$

From left to right, the terms of (1) represent the total derivative of the three-dimensional wind field $\vec{v}$, the Coriolis force, the pressure gradient force, effective gravity, and friction. The total derivative in the first term of (1) has the general form:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\hat{\mathbf{i}} + v\frac{\partial}{\partial y}\hat{\mathbf{j}} + w\frac{\partial}{\partial z}\hat{\mathbf{k}}$$

Equation (1) may be expanded into its component forms and recast into spherical coordinates. Presented without derivation, the corresponding equations of motion (for *u* and *v* only) are:

$$\frac{Du}{Dt} - \frac{uv\tan\phi}{a} + \frac{uw}{a} - fv + 2\Omega w\cos\phi = -\frac{1}{\rho}\frac{\partial p}{\partial x} + F_{rx} \quad (2a)$$

$$\frac{Dv}{Dt} + \frac{u^2\tan\phi}{a} + \frac{vw}{a} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} + F_{ry} \quad (2b)$$

where ϕ = latitude, Ω = the rotation rate of the Earth (7.292 x 10^{-5} s^{-1}), u = zonal component of the wind, v = meridional component of the wind, w = vertical component of the wind, p = pressure, ρ = density, $f = 2\Omega \sin \phi$ = Coriolis parameter, and a = Earth's radius (6.378 x 10^6 m).

The second and third terms of (2) represent terms related to the curvature of the Earth. The fourth term of (2b) and fourth and fifth terms of (2a) are Coriolis terms. The two terms on the right-hand side of (2) are pressure gradient and frictional terms, respectively. These equations, alongside the corresponding equation for *w*, define the equations of motion. We note that these forms of the equations of motion are presented with height, or *z*, as the vertical coordinate.

Scale Analysis of the Equations of Motion

The equations of motion, as stated in their full form in (2) above, are quite complex. They describe all types and scales of atmospheric motions, including some (e.g., acoustic waves) that are either negligible or altogether irrelevant for the study of synoptic-scale motions.

We desire to simplify (2) into a form that reflects only the most important processes on the synoptic-scale. In other words, we want to keep only those terms that are large for synoptic-scale motions. We do so by performing a *scale analysis* on the terms in (2). This is done by replacing each variable with an appropriate characteristic value for that variable based upon observed values for mid-latitude synoptic-scale motions. The characteristic values appropriate for the scale analysis of (2) are given by Table 1 below.

Variable	**Characteristic Value**	**Description**
u, v	$U \approx 10$ m s^{-1}	Horizontal velocity scale
w	$W \approx 0.01$ m s^{-1}	Vertical velocity scale
x, y	$L \approx 10^6$ m	Horizontal length scale
z	$H \approx 10^4$ m	Depth scale
$\delta p/\rho$	$\delta P/\rho \approx 10^3$ m^2 s^{-2}	Horizontal pressure fluctuation scale
t	$L/U \approx 10^5$ s	Time scale
f, $2\Omega \cos \phi$	$f_0 \approx 10^{-4}$ s^{-1}	Coriolis scale

Table 1: Characteristic values appropriate for synoptic-scale motions for the variables found within Equation (2).

Several important insights into what is meant by 'synoptic-scale motions' can be drawn from the values in the table above:

- The magnitude of the horizontal velocity is much larger – several orders of magnitude – than the magnitude of the vertical velocity.
- Synoptic-scale features have horizontal extent of hundreds of kilometers or more.
- These features also extend through a meaningful depth (~10 km) of the troposphere.
- Synoptic-scale motions evolve slowly, on the order of 1 day (8.64×10^4 s) or longer.
- We are considering only mid-latitude phenomena by stating that $f_0 \approx 10^{-4}$ s^{-1}, which is its value at latitude $\phi = 45°$N.

If we plug in the appropriate characteristic values from the table above into (2), we obtain:

u-mom.	$\frac{Du}{Dt}$	$-fv$	$2\Omega w \cos\phi$	$\frac{uw}{a}$	$-\frac{uv\tan\phi}{a}$	$-\frac{1}{\rho}\frac{\partial p}{\partial x}$	F_{rx}
v-mom.	$\frac{Dv}{Dt}$	Fu		$\frac{vw}{a}$	$\frac{u^2\tan\phi}{a}$	$-\frac{1}{\rho}\frac{\partial p}{\partial y}$	F_{ry}
Scale	U^2/L	$f_0 u$	$f_0 w$	UW/a	U^2/a	$\delta P/\rho L$	DU/H^2
Value (m s^{-2})	10^{-4}	10^{-3}	10^{-6}	10^{-8}	10^{-5}	10^{-3}	10^{-12}

Table 2: Scale analysis of the equations of motion.

Note that the D in the scaling of the frictional terms is the *eddy diffusivity*, a term related to turbulent (or frictional) processes within the boundary layer. It is of order $D \approx 10^{-5}$ m^2 s^{-1}.

The Geostrophic Approximation on Constant Height Surfaces

From Table 2 above, it is apparent that for synoptic-scale motions, two terms are at least one order of magnitude larger than the other terms. These are the Coriolis and pressure gradient terms. If we retain only these two terms in (2), the following expressions result:

$$-fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{3a}$$

$$fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} \tag{3b}$$

Equation (3) represents the *geostrophic relationship*. The geostrophic relationship gives the *approximate* relationship between the horizontal pressure gradient and the horizontal velocity (as scaled by the Coriolis parameter) in large-scale extratropical/mid-latitude weather systems.

Geostrophic balance is defined as the balance between the horizontal pressure gradient and Coriolis forces. This means that the horizontal pressure gradient force and Coriolis force are of equal magnitude to but directed in opposite directions from each other. The horizontal pressure gradient force *always* points from high toward low pressure, in contrast to the horizontal pressure gradient itself, which always points from low toward high pressure. Under the constraint of geostrophic balance, the Coriolis force points in the opposite direction of the horizontal pressure gradient force, or from low toward high pressure. This balance is depicted in Figure 1 below.

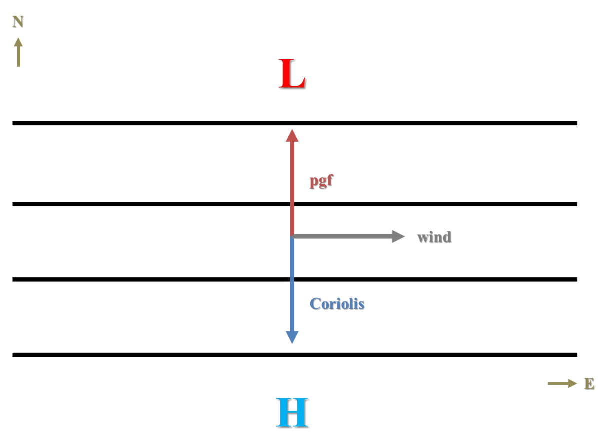

Figure 1. Graphical depiction of geostrophic balance for an idealized example in the Northern Hemisphere with low pressure to the north and high pressure to the south. The pressure gradient force (pgf) is denoted in red, the Coriolis force is denoted in blue, and the resultant wind is denoted in grey. Black lines denote idealized isobars, here depicted on a constant height surface.

The magnitude of the horizontal pressure gradient force is directly proportional to the magnitude of the horizontal pressure gradient, as one might expect from inspection of the right-hand side of (3). Similarly, the magnitude of the Coriolis force is directly proportional to the horizontal wind speed (i.e., magnitude of the horizontal velocity vector), as one might expect from inspection of the left-hand side of (3). Thus, a larger horizontal pressure gradient requires a larger Coriolis force to balance it, thus necessitating a faster horizontal wind speed.

Note that in the Northern Hemisphere, the Coriolis force always points perpendicular and to the right of the wind, as depicted in Figure 1 above. We can prove this from (2). If we retain only the total derivative terms and terms that involve the Coriolis parameter *f* in (2), we obtain:

$$\frac{Du}{Dt} = fv \qquad (4a)$$

$$\frac{Dv}{Dt} = -fu \qquad (4b)$$

The total derivative terms on the left-hand side of (4) represent *accelerations*; i.e., changes in u and v with time following the motion. Thus, (4) represents parcel accelerations due exclusively to the Coriolis force. Note that in the Northern Hemisphere, f is always positive.

Consider air that is moving from south to north, such that $v > 0$. In this case, for $f > 0$, the left-hand side of (4a) must be positive. This means that, following the motion, the air parcel is accelerating to the east ($u > 0$). This is 90° to the right of the air's motion from south to north. Similar arguments can be made for any given wind direction. Thus, the Coriolis force is always directed 90° to the right of the wind in the Northern Hemisphere.

How is Geostrophic Balance Achieved?

Consider air that is initially at rest. Because the Coriolis force is directly proportional to the wind speed, the Coriolis force is initially zero. Now consider there to be a horizontal pressure gradient on the synoptic-scale, such as is depicted in Figure 2 with lower pressure to the north and higher pressure to the south. In this case, the horizontal pressure gradient force is directed from south to north. This causes the air to accelerate and, thus, begin to move. This brings about a force imbalance. Initially, as the horizontal pressure gradient force is the only force acting upon the air, it moves from high toward low pressure – i.e., to the north. However, as the air begins to move, the Coriolis force becomes non-zero. Very quickly, on the time span of perhaps minutes, the horizontal pressure gradient force and Coriolis force come into (near-)balance, and the air is deflected to the right.

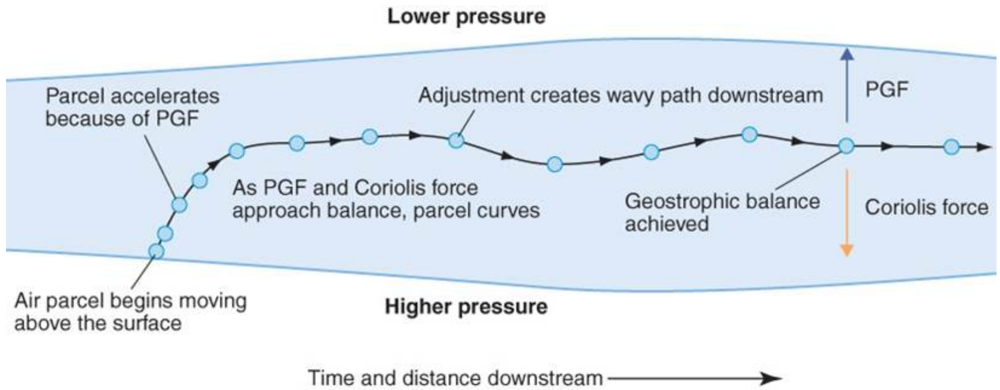

Figure 2. Schematic illustrating how geostrophic balance is initially achieved, what happens when geostrophic balance is disrupted, and the restoration of geostrophic balance through geostrophic adjustment. Figure obtained from Figure 6-17 of *Meteorology: Understanding the Atmosphere*, 4^{th} Ed., S. Ackerman and J. Knox.

Consider air that is moving but that obeys geostrophic balance. It is possible for geostrophic balance to be disrupted, such as by localized heating, deflection of the flow by terrain features, and/or localized instabilities within the flow. The process by which geostrophic balance is restored is known as *geostrophic adjustment*. Simply put, during geostrophic adjustment, both the wind and pressure (and, by extension, mass) fields adjust to one another such that balance may again be achieved. The wavy path that the air takes in Figure 2 after it initially achieves geostrophic balance is an example of geostrophic adjustment; the horizontal pressure gradient and Coriolis forces adjust to each other, resulting in a temporarily wavy flow until balance is restored. Balance restoration is typically associated with gravity and/or inertia-gravity waves, the characteristics and dynamics of which are beyond the scope of this class.

The Geostrophic Relationship on Isobaric Surfaces

To obtain (3), we conducted a scale analysis of the equations of motion applicable on constant height surfaces. Consequently, (3) is applicable only on constant height surfaces. We wish to now obtain a form of (3) that is applicable on isobaric surfaces. To do so, a coordinate transformation from the z to the p vertical coordinate is necessary. First, consider an idealized example of two isobaric surfaces that vary in height only in the x-direction, as seen in Figure 3.

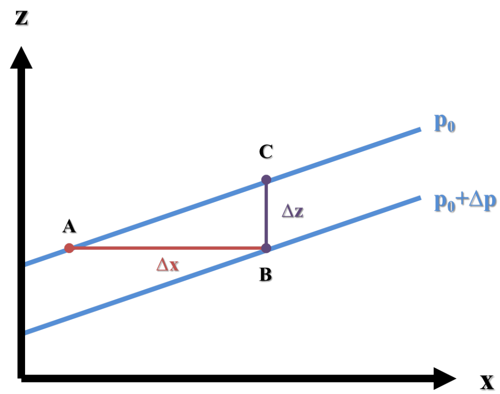

Figure 3. Idealized depiction of two isobaric surfaces, p_0 and $p_0+\Delta p$, that vary in height only within the *x*-direction. The distances Δx and Δz represent the distance between these two isobaric surfaces in the *x*- and *z*- directions, respectively.

Consider the change in pressure as one moves from point A, where $p = p_0$, to point B, where $p = p_0 + \Delta p$. This can be represented in terms of a partial derivative, where:

$$p_B - p_A = \left(\frac{\partial p}{\partial x}\right)_z \Delta x \tag{5a}$$

The subscript *z* in (5a) denotes that this partial derivative is evaluated on a constant height surface, which is appropriate because $z_B = z_A$ here. If one were to apply a centered finite difference approximation halfway between point A and point B to the partial derivative in (5a), the right-hand side of (5a) would simply become Δp, equal to p_B ($p_0 + \Delta p$) minus p_A (p_0).

Likewise, consider the change in pressure as one moves from point B, where $p = p_0 + \Delta p$, to point C, where $p = p_0$. This can also be represented in terms of a partial derivative, where:

$$p_C - p_B = \left(\frac{\partial p}{\partial z}\right)_x \Delta z \tag{5b}$$

The subscript *x* in (5b) denotes that this partial derivative is evaluated at a constant value of *x*, which is appropriate because $x_B = x_C$ here. If one were to apply a centered finite difference approximation halfway between point B and point C to the partial derivative in (5b), the right-hand side of (5b) would simply become -Δp, equal to p_C (p_0) minus p_B ($p_0 + \Delta p$).

Because $p_A = p_C$, we can substitute p_A for p_C in (5b). If we then multiply the resulting equation by -1, an expression for $p_B - p_A$ results. Equating this to (5a), we obtain:

$$\left(\frac{\partial p}{\partial x}\right)_z \Delta x = -\left(\frac{\partial p}{\partial z}\right)_x \Delta z \tag{6}$$

If we divide (6) by Δx, we obtain:

$$\left(\frac{\partial p}{\partial x}\right)_z = -\left(\frac{\partial p}{\partial z}\right)_x \frac{\Delta z}{\Delta x} \tag{7}$$

Upon inspection of Figure 3, we can see that Δz/Δx is equivalent to the slope of an isobaric surface. Taking the limit of this term in (7) as Δx approaches 0 and applying the definition of the partial derivative to the result, we obtain:

$$\left(\frac{\partial p}{\partial x}\right)_z = -\left(\frac{\partial p}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_p \tag{8}$$

In (8), the subscript of p indicates that the partial derivative is evaluated on an isobaric surface. If we substitute into (8) with the hydrostatic equation, we obtain:

$$\left(\frac{\partial p}{\partial x}\right)_z = \rho g \left(\frac{\partial z}{\partial x}\right)_p \tag{9a}$$

If we were to repeat the same process, except for pressure varying only in the y-direction, we would obtain:

$$\left(\frac{\partial p}{\partial y}\right)_z = \rho g \left(\frac{\partial z}{\partial y}\right)_p \tag{9b}$$

It is the expressions in (9) that enable us to convert the pressure gradient terms evaluated on a constant height surface (from Equation 3) to their equivalent height gradient terms evaluated on an isobaric surface. If we plug (9) into (3), we obtain:

$$fv = g\frac{\partial z}{\partial x} \tag{10a}$$

$$fu = -g\frac{\partial z}{\partial y} \tag{10b}$$

Or, making use of the definition of the geopotential that we introduced in an earlier lecture, (10) can be written as:

$$fv = \frac{\partial \Phi}{\partial x} \tag{11a}$$

$$fu = -\frac{\partial \Phi}{\partial y} \tag{11b}$$

Note that there is no reference to time in (3), (10), or (11). Thus, the geostrophic relationship is a *diagnostic relationship*; it can only be used to diagnose the horizontal velocity as a function of the pressure or height field at a given time. This stands in contrast to a *prognostic relationship*, or one that involves a reference to time (such as through the presence of a partial derivative with respective to time), which permits the prediction of the evolution of the velocity field.

The Geostrophic Wind

To express geostrophic balance, u in (3), (10), and (11) is replaced by u_g while v is replaced by v_g. These two variables, u_g and v_g, define what is known as the *geostrophic wind*. The geostrophic wind is a stable, slowly-evolving flow. Contrast this with highly curved, rapidly accelerating, or near-surface flows, which can evolve more rapidly or may be unstable. Given that density is not routinely measured and that most meteorological analysis is conducted on isobaric surfaces, either (10) or (11) are used to evaluate the geostrophic wind from observations.

Because the pressure gradient and Coriolis forces are of equal magnitude but in opposite directions under the constraint of geostrophic balance, the geostrophic wind blows *parallel* to the isobars on a given height surface. This is depicted in Figure 1 above. The geostrophic wind similarly blows parallel to lines of constant geopotential height on an isobaric surface. This relationship between the wind and the isobars or constant height contours is one of the defining characteristics of geostrophic balance. An application of this relationship is the *Buys-Ballot law*, which states that when the (geostrophic) wind is hitting your back, low pressure is to your left.

In the mid-latitudes, the geostrophic wind approximates the true horizontal velocity to within ~10%. How closely the wind is to being parallel to the isobars or constant height contours on a synoptic analysis provides us a measure for how close the atmosphere is to satisfying geostrophic balance at that location. Departures from geostrophic balance occur most commonly when the centrifugal and/or frictional forces are non-negligible. The former is associated with curved flow, whereas the latter is important primarily near Earth's surface within the planetary boundary layer. We will consider how and when the wind is influenced by these forces in later lectures.

The Ageostrophic Wind

In defining the geostrophic relationship above, we neglected the total derivative terms on the left-hand side of (2). We now want to revisit (2), keeping these terms (as well as the Coriolis and pressure gradient terms) while continuing to neglect the rest. The resultant equations are:

$$\frac{Du}{Dt} = fv - \frac{1}{\rho}\frac{\partial p}{\partial x} \tag{12a}$$

$$\frac{Dv}{Dt} = -fu - \frac{1}{\rho}\frac{\partial p}{\partial y} \tag{12b}$$

However, (3) – with u_g and v_g substituted for u and v – gives expressions for the pressure gradient terms that appear in (12). If we substitute for those terms into (12), we obtain:

$$\frac{Du}{Dt} = fv - fv_g = f(v - v_g) \tag{13a}$$

$$\frac{Dv}{Dt} = -fu + fu_g = -f(u - u_g) \tag{13b}$$

The terms $v - v_g$ and $u - u_g$ define the *ageostrophic wind*, where $v - v_g = v_{ag}$ and $u - u_g = u_{ag}$. What does (5) mean? Most simply, acceleration is tied to the ageostrophic wind, or departure of the total wind from the geostrophic wind. Acceleration is important on the synoptic-scale primarily in the vicinity of jet streams and jet streaks. Because the geostrophic wind approximates the full wind to within ~10%, the ageostrophic wind is said to be one order of magnitude smaller than the geostrophic wind. Given a typical scaling of $U \sim 10$ m s^{-1} on the synoptic scale, the characteristic scale for the ageostrophic wind is ~1 m s^{-1}.

The Rossby Number

In our earlier scale analysis, we found that the acceleration terms (the total derivatives) are typically one order of magnitude smaller than the Coriolis and pressure gradient terms. Just how close this is to being true for any given weather situation can be assessed by determining the ratio between the characteristic scales of the acceleration and Coriolis terms. This defines the *Rossby number*, named after the famous meteorologist Carl-Gustav Rossby, and is given by:

$$Ro = \frac{\frac{U^2}{L}}{f_o U} = \frac{U}{f_0 L} \tag{14}$$

Note that the U, f_0, and L in (14) are not the same as their values in Table 1. Rather, they represent values specific to the synoptic-scale weather conditions being assessed.

For $Ro \approx 0.1$, the magnitude of the acceleration term is one order of magnitude smaller than the magnitude of the Coriolis term. This describes geostrophic balance. For $Ro \approx 1$ or $Ro > 1$, the magnitude of the acceleration term is comparable to or exceeds the magnitude of the Coriolis term. In such situations, geostrophic balance does *not* hold. Instead, we must consider gradient wind balance or cyclostrophic balance for such situations.

For Further Reading

Section 1.3 of *Midlatitude Synoptic Meteorology* by G. Lackmann introduces geostrophic balance and the Rossby number. Section 3.2 of *Mid-Latitude Atmospheric Dynamics* by J. Martin provides a discussion of the geostrophic approximation and geostrophic balance. Section 4.1 of *Mid-Latitude Atmospheric Dynamics* provides a generalized discussion of how equations posed on constant height surfaces may be transformed into equations posed on isobaric surfaces.