Synoptic Meteorology I: Divergence and Vertical Motion

4 November 2014

Why Do We Care About Vertical Motion?

We care about vertical motion for several reasons…

- Condensation, and thus cloud and precipitation formation, generally occurs as a result of cooling an air parcel (or layer) to saturation as a result of ascent.
- As we will demonstrate next semester, mid-latitude cyclone and anticyclone formation, decay, and movement are functions of patterns of ascent and descent.
- Vertical vorticity amplification and generation are related to vertical motion and its horizontal gradients. We will explore this in part next semester as well.

One might ask, how do we evaluate vertical motion, whether qualitatively or quantitatively? In this lecture, we consider one method of doing so, relating vertical motion to divergence. We will explore other methods for doing so in greater detail next semester.

Divergence Calculation Methods

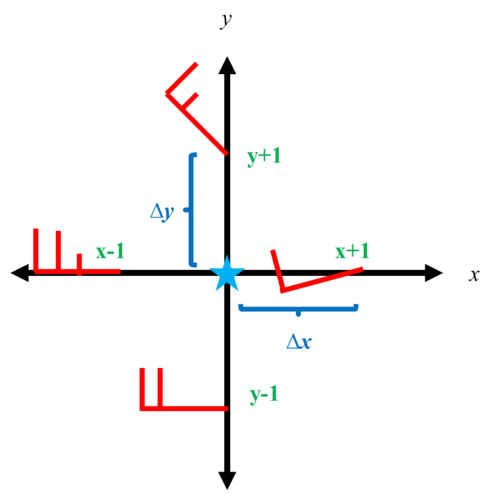
In our previous lecture, we defined divergence as:

$$\delta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \tag{1a}$$

$$\delta = \frac{\partial V}{\partial s} - V\frac{\partial \alpha}{\partial n} \tag{1b}$$

where (1a) is divergence represented in Cartesian coordinates and (1b) is divergence represented in natural coordinates. Given at least four wind observations – two each along the x- and y- or s- and n-axes – one could use centered finite difference approximations to compute divergence. This is illustrated in a Cartesian coordinate system in Figure 1.

The process of evaluating divergence in this way can be cumbersome, however, because of the need to first convert each wind observation to its u and v components. Evaluating divergence using natural coordinates can help overcome this hurdle but is best for qualitative rather than quantitative purposes. Furthermore, because we rarely have observations over a uniform grid as in Figure 1, we often have to interpolate data onto such a grid to calculate divergence using finite differences. Finally, recall that finite differences are approximations by their very nature and are formally valid only over finite distances (here, ∂x and ∂y). It is up to the analyst to determine whether these limitations are acceptable or not for the case being considered.



$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\approx\frac{u_{x+1}-u_{x-1}}{2\Delta x}+\frac{v_{y+1}-v_{y-1}}{2\Delta y}$$

Figure 1. Conceptual example of evaluating divergence at the location given by the blue star using a centered finite difference and winds (half-flag: 5 kt, flag: 10 kt, pennant: 50 kt) from four surrounding observing locations, located at points *x-1*, *x+1*, *y-1*, and *y+1*. The distances Δx and Δy are indicated by the blue text and brackets and, formally, are in meters.

Alternatively, one could use an integral-based method to calculate the divergence into or out of any desired area. For any arbitrary two-dimensional region, the divergence within that region is defined (from calculus) as:

$$\delta=\frac{1}{A}\lim_{A\to 0}\iint_A \nabla\cdot\vec{v}\,da \tag{2}$$

In (2), A denotes the area of the two-dimensional region, and the double integral represents integration about the region with area A. Omitting the limit, (2) can be approximated as:

$$\delta\approx\overline{\nabla\cdot\vec{v}}=\frac{1}{A}\iint_A \nabla\cdot\vec{v}\,da \tag{3}$$

The overbar in (3) indicates a quantity that is averaged over the area A of the region. If we apply Green's theorem to (3), we obtain:

$$\delta=\frac{1}{A}\int_L v_n\,dl \tag{4}$$

In (4), L represents the boundary of the region with area A, dl represents a finite line segment along the boundary L, and v_n represents the component of the wind normal (perpendicular) to the boundary L (where $v_n > 0$ for outward-directed flow and $v_n < 0$ for inward-directed flow). Approximating the integral in (4) with a summation, we obtain:

$$\delta \approx \frac{1}{A}\sum_{l \to L} v_n \Delta l \tag{5}$$

Equation (5) states that if we know the component of the wind normal to our region at a number of different locations along the boundary L of that region, along with the distance Δl between locations and the area A of the region, we can compute the mean divergence within the region. This calculation works best when many observations – such as obtained from a model grid – are available. A conceptual example of how this may be done qualitatively is presented in Figure 2.

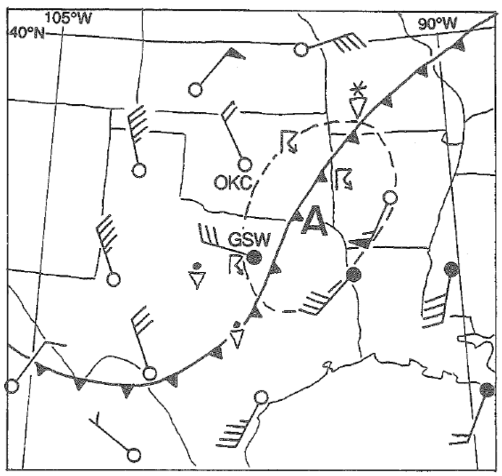


Figure 2. 850 hPa wind (barbs; half-flag: 5 kt, flag: 10 kt, pennant: 50 kt), surface frontal boundary locations (solid triangled line), and sensible weather (symbols) at 0000 UTC 27 January 1969. The dashed region surrounding location A is the region considered in the discussion below. Figure reproduced from *Weather Analysis* by D. Djurić, their Figure 4-8.

Consider the portion of the dashed region located to the east of the surface cold front. We have two wind observations, both roughly aligned parallel to the dashed line. The normal component of the wind at these locations, then, is zero. Given that the observations imply that the 850 hPa is out of the southwest at 40-55 kt across this entire area, there is a large inward-directed normal component of the wind on the southern edge of the dotted region. This is balanced by a large outward-directed normal component of the wind on the north-northeastern flank of the dotted region, however.

Now consider the portion of the dashed region located to the west of the surface cold front. We have two wind observations, both roughly aligned normal to the dashed line. As both are directed inward, we have a negative contribution to our summation in (5). We can infer from the wind observation in northeastern Kansas that the wind is likely roughly parallel to the dotted region on its far northern extent. Thus, adding all of our contributions together, we qualitatively assess that the mean divergence within the dotted region is negative, implying convergence. If we so desired, we could explicitly calculate this using (5) by interpolating the wind observations to the edge of the dotted region, compute the normal component of the wind at each location, and determine the distance along the perimeter of the dotted region between observations.

There do exist other methods of computing divergence, such as the Bellamy method for computing divergence at a location given wind observations at three nearby locations. As a result, the above is not intended to be a comprehensive listing of how divergence may be calculated. It is sufficient for our purposes, however, and enables us to move forward to relating divergence and vertical motion.

The Relationship Between Divergence and Vertical Motion

The continuity equation, expressed in isobaric coordinates, is given by:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial \omega}{\partial p}=0 \tag{6}$$

Equation (6) states that atmospheric mass is conserved. For our purposes, we will proceed as though this is always the case. However, interested readers should consult Section 1.3 of *Midlatitude Synoptic Meteorology* by G. Lackmann for a discussion of exceptions to this rule.

The first two terms on the left-hand side of (6) should be familiar to us by now: they represent the divergence on an isobaric surface. If we rewrite (6) with this in mind, we obtain:

$$\delta=-\frac{\partial \omega}{\partial p} \tag{7}$$

Note that we have moved the remaining partial derivative term to the right-hand side of the equation. Equation (7) states that a given location, the negative of the partial derivative of vertical motion ω with respect to pressure p is equal to the divergence. Let us now integrate (7) between two arbitrary isobaric surfaces p_b and p_t, where $p_b > p_t$ (i.e., p_b is found closer to the surface than is p_t given that pressure decreases with increasing altitude). If we do so, we obtain:

$$\int_{p_b}^{p_t} \delta dp=-\int_{p_b}^{p_t} \frac{\partial \omega}{\partial p} dp \tag{8}$$

The right-hand side of (8) can be approximated as $-\int_{p_b}^{p_t} \partial \omega$, which is equal to $-(\omega(p_t) - \omega(p_b))$. If we substitute this in to (8), we obtain:

$$\int_{p_b}^{p_t} \delta dp = -(\omega(p_t) - \omega(p_b)) \tag{9}$$

Equation (9) makes clear the relationship between divergence and vertical motion. Specifically, the difference in vertical motion ω over some vertical layer bounded by two isobaric levels p_b and p_t (where $p_b > p_t$) is equal to the vertically-integrated divergence within that layer.

We can continue to operate on (9) if we desire. If we replace δ by $\overline{\delta}$, the pressure-weighted average divergence within the layer between p_b and p_t, then the left-hand side of (9) can be approximated as $\overline{\delta}\int_{p_b}^{p_t} dp$, which is equal to $\overline{\delta}[p_t - p_b]$. Substituting, we obtain:

$$\overline{\delta}[p_b - p_t] = \omega(p_t) - \omega(p_b) \tag{10}$$

Note that we have moved the leading negative from the right-hand side to the left-hand side of the equation, allowing us to flip the order of the subtraction. Equation (10) states that the difference in vertical motion over a vertical layer bounded by p_t and p_b is equal to the product of the layer-mean divergence and the difference in pressure over the vertical layer.

In this discussion, we have attributed the divergence to the divergence of the total wind. However, recall that to good approximation, we stated that the geostrophic wind is non-divergence. Thus, the divergence in the foregoing discussion can be viewed as equal to the divergence of the ageostrophic wind.

Dines' Compensation and the Level of Non-Divergence

Derivation

Let us now consider a hypothetical atmosphere comprised of two layers: one between the surface (p_{sfc}) and some middle tropospheric isobaric level (p_L), and one between some middle tropospheric isobaric level (p_L) and the tropopause (p_{trop}). This is depicted in Figure 3 below.

Figure 3. Hypothetical atmosphere comprised of two layers, as described in the text above.

Let us apply (9) for each of these two layers. For the layer lower, where $p_b = p_{sfc}$ and $p_t = p_L$, we obtain:

$$\int_{p_{sfc}}^{p_L} \delta dp = -\left(\omega\left(p_L\right) - \omega\left(p_{sfc}\right)\right) \tag{11a}$$

The surface is a rigid bound on vertical motions; right at the surface, ω is equal to zero. Thus, $\omega(p_{sfc}) = 0$, such that (11a) simplifies to:

$$-\int_{p_{sfc}}^{p_L} \delta dp = \omega\left(p_L\right) \tag{11b}$$

For the upper layer, where $p_b = p_L$ and $p_t = p_{trop}$, we obtain:

$$\int_{p_L}^{p_{trop}} \delta dp = -\left(\omega\left(p_{trop}\right) - \omega\left(p_L\right)\right) \tag{12a}$$

The tropopause, like the surface, is a rigid bound on vertical motions; right at the tropopause, outside of intense micro- to mesoscale vertical motions such as in thunderstorms, ω is equal to zero. Thus, $\omega(p_{trop}) = 0$, such that (12a) simplifies to:

$$\int_{p_L}^{p_{trop}} \delta dp = \omega\left(p_L\right) \tag{12b}$$

Upon inspection, it is clear that we have two separate equations for $\omega(p_L)$, as given by (11b) and (12b). If we equate these equations, we obtain:

$$-\int_{p_{sfc}}^{p_L} \delta dp = \int_{p_L}^{p_{trop}} \delta dp \qquad (13)$$

In other words, the vertically-integrated divergence in the lower layer is cancelled out by the vertically-integrated divergence in the upper layer. Stated differently, the divergence within the lower layer is equal in magnitude and opposite in sign to the divergence in the upper layer. This implies that the two are in balance with each other, such that one compensates for the other. This important principle is known as *Dines' compensation principle*.

The sign of divergence must change at least once between p_{sfc} and p_{trop} in order for (13) to be true. Thus, an important corollary to Dines' compensation principle states that there must be at least one level at which the divergence is equal to zero. This level is known as the *level of non-divergence*. In the troposphere, we often find a level of non-divergence in the middle troposphere, typically between 500-600 hPa. This is why (in part, at least, without further delving into atmospheric dynamics) either of these standard isobaric levels are typically understood to be the *steering level* for synoptic-scale mid-latitude weather systems.

Application

Let us consider a couple of qualitative examples. First, consider the case where there is convergence within the lower layer. From Dines' compensation, we know that this must be balanced by divergence within the upper layer. We also know that there must be a level of non-divergence at the interface between the two layers, or at p_{L}. We can use this information to obtain the sign of ω at p_{L}. Given $\delta < 0$ within the lower layer, $p_{\text{L}} < p_{\text{sfc}}$, and a leading negative sign on the entire expression, we know that the left-hand side of (11b) – and thus $\omega(p_{\text{L}})$ – will be negative. This indicates rising motion, or ascent, at p_{L}. Likewise, given $\delta > 0$ within the upper layer and $p_{\text{trop}} < p_{\text{L}}$, we know that the right-hand side of (12b) – and thus $\omega(p_{\text{L}})$ again – will be negative. This provides a sanity check on our earlier solution. Vertical profiles of divergence and vertical motion accompanying this example are provided in Figure 4.

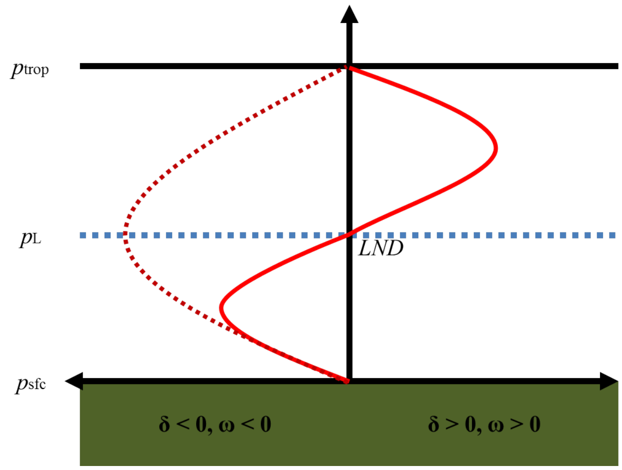


Figure 4. Vertical profiles of divergence (solid red line) and vertical motion (dashed red line) corresponding to the case of convergence in the lower layer, divergence in the upper layer, and the level of non-divergence at the interface between the two layers. Note how ascent ($\omega < 0$) is maximized at the level of non-divergence (*LND*).

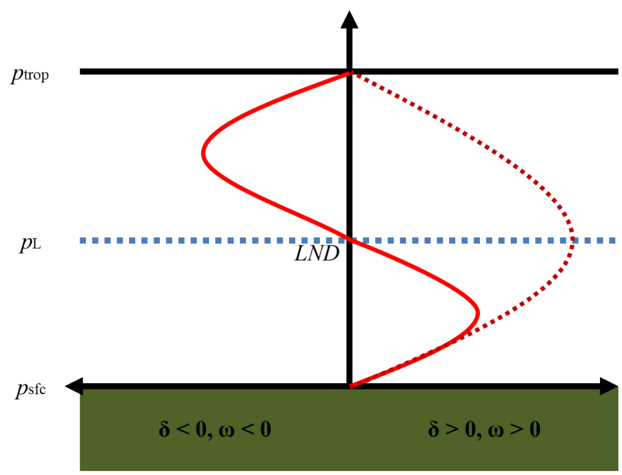


Figure 5. Vertical profiles of divergence (solid red line) and vertical motion (dashed red line) corresponding to the case of divergence in the lower layer, convergence in the upper layer, and the level of non-divergence at the interface between the two layers. Note how descent ($\omega > 0$) is maximized at the level of non-divergence (*LND*).

Next, consider the case where there is divergence within the lower layer. From Dines' compensation, we know that this must be balanced by convergence within the upper layer. We also know that there must be a level of non-divergence at the interface between the two layers, or at p_L. We can use this information to obtain the sign of ω at p_L. Given $\delta > 0$ within the lower layer, $p_L < p_{sfc}$, and a leading negative sign on the entire expression, we know that the left-hand side of (11b) – and thus $\omega(p_L)$ – will be positive. This indicates sinking motion, or descent, at p_L.

Likewise, given $\delta < 0$ within the upper layer and $p_{trop} < p_L$, we know that the right-hand side of (12b) – and thus $\omega(p_L)$ again – will be positive. This provides a sanity check on our earlier solution. Vertical profiles of divergence and vertical motion accompanying this example are provided in Figure 5.

The real atmosphere can typically not be considered simply in the context of two vertical layers. However, it is relatively straightforward to evaluate the vertical profile of vertical motion given patterns of divergence even in quite complex situations. Let us start with equation (9), which we repeat below for simplicity:

$$\int_{p_b}^{p_t} \delta dp = -\left(\omega(p_t) - \omega(p_b)\right)$$

If we let $p_b = p_{sfc}$, then $\omega(p_{sfc}) = 0$. Thus, the above equation becomes:

$$\omega(p_t) = -\int_{p_{sfc}}^{p_t} \delta dp \tag{14}$$

In other words, the vertical motion at any isobaric level p_t is equal to the negative of the integrated divergence between the surface and p_t. Thus, given a vertical profile of divergence, we can start at the surface and integrate upward – whether qualitatively or quantitatively – to obtain the corresponding vertical profile of vertical motion. Examples are presented in Figures 6 and 7.

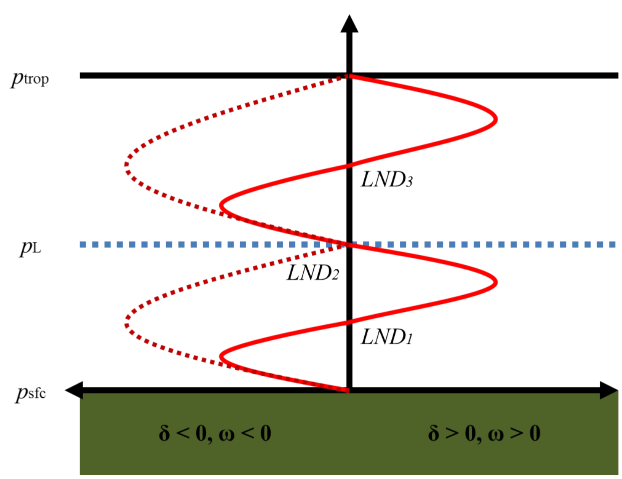


Figure 6. Vertical profiles of divergence (solid red line) and vertical motion (dashed red line) corresponding to the case of equal areas of convergence-divergence-convergence-divergence from the surface to the tropopause. Note how ascent is either maximized or is zero at a level of non-divergence, depending upon the vertical integral of δ from the surface to that altitude.

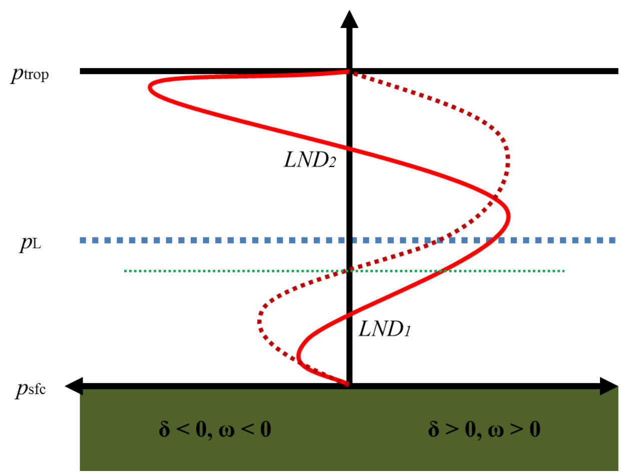


Figure 7. Vertical profiles of divergence (solid red line) and vertical motion (dashed red line) corresponding to the case of unequal areas of convergence-divergence-convergence from the surface to the tropopause. Again note the relationship between ω and levels of non-divergence, somewhat more complicated than in the previous examples.

In Figure 6, we are presented with an example that is similar to Figure 4. Starting at the surface and working upward, we begin with convergence and, thus, ascent. When the vertical integral (or *summation*) of divergence reaches its maximum negative value – where the vertical profile of divergence switches from convergence to divergence, the first level of non-divergence – ascent is maximized. Ascent decreases to zero when the vertical integral of divergence becomes zero at p_L, as the area of convergence near the surface becomes exactly balanced by the area of divergence above it. The pattern repeats as you continue upward from p_L.

In Figure 7, we are presented with an even more complex – but also more realistic – example. Just above the surface, we find weak convergence that extends over a shallow vertical layer. Ascent peaks at the lower of the two levels of non-divergence and becomes zero at the dotted green line, where the vertically-integrated divergence (area beneath the solid red curve) becomes equal to zero. Divergence is found from the lower to the upper troposphere, and consequently descent over a deep vertical layer develops. This descent is maximized at the second level of non-divergence, where the vertical integral of divergence is at a positive maximum, and then decreases to zero at the tropopause as the vertical integral of divergence approaches zero given the shallow layer of strong convergence just below the tropopause.

To this point, we have demonstrated the relationship between vertical motion and divergence. In our next set of lectures, we strive to understand thermodynamic limiters on vertical motion (and thus, by extension, divergence). Next semester, we will consider several additional means of evaluating vertical motion from synoptic charts and explicitly link vertical motion to both cyclone and anticyclone development and vertical vorticity development.

For Further Reading

The isobaric form of the continuity equation is presented in Section 1.3 of *Midlatitude Synoptic Meteorology* by G. Lackmann, among other references. The evaluation of divergence using integral-based methods is discussed in Section 4-4 of *Weather Analysis* by D. Djurić. The relationship between divergence and vertical motion is discussed in Section 4-5 of *Weather Analysis*. A derivation of the mass continuity equation and its basic application to vertical motion is also provided by Section 4.1 of *Mid-Latitude Atmospheric Dynamics* by J. Martin.