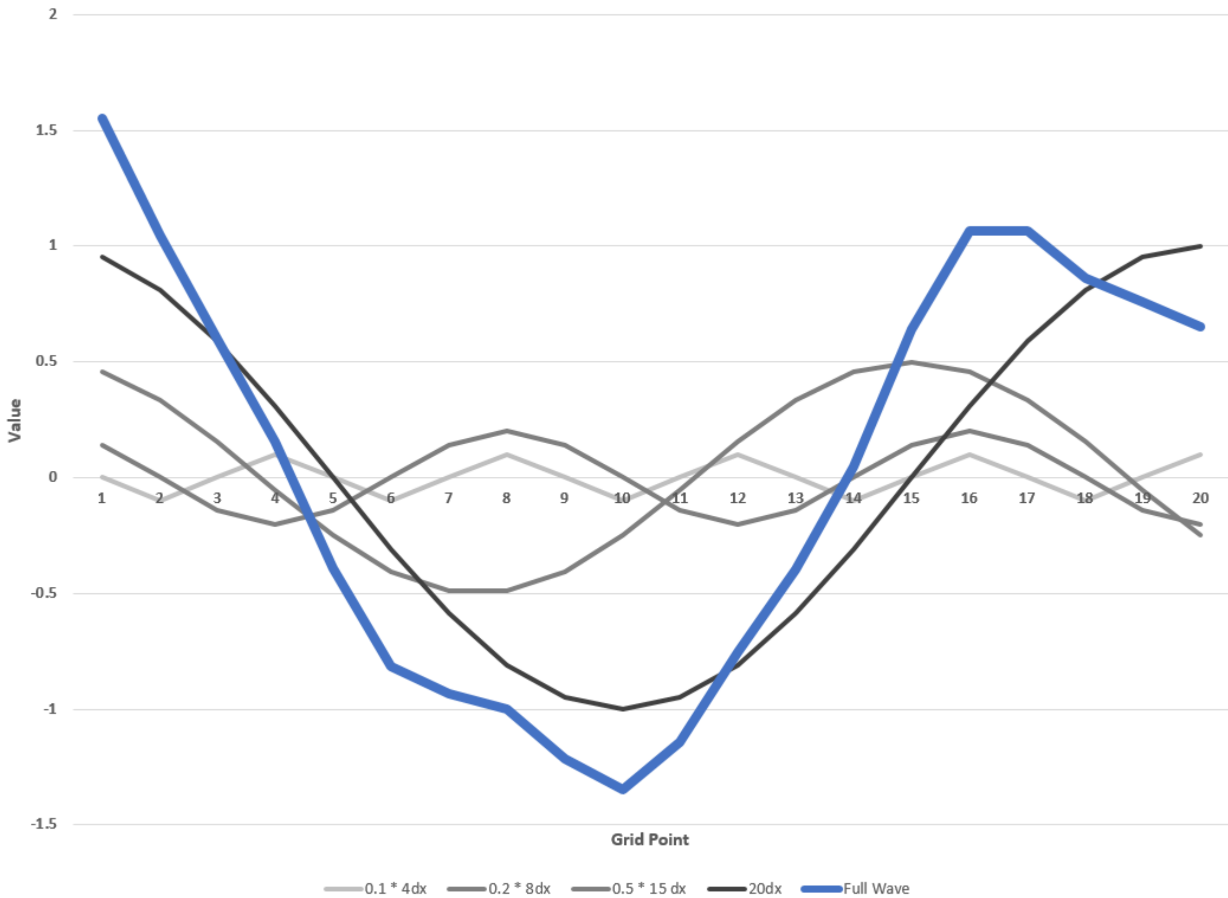
Clarifying Damping/Diffusion and Numerical Dispersion

It can be challenging to conceptually understand the differences between damping/diffusion and numerical dispersion, since the effects of each can appear similar in practice. This document is an attempt to try to clarify the differences between the two concepts.

Consider the following graph:



There are five waves depicted here, each of which have the general form $A \cos kx$, where A is an amplitude, k is a wavenumber equal to $2\pi/L$, L is wavelength = $n\Delta x$, and x is position. Here, we let Δx = 10,000 m. Four waves are plotted in various shades of grey: a $4\Delta x$ wave with $A = 0.1$, an $8\Delta x$ wave with $A = 0.2$, a $15\Delta x$ wave with $A = 0.5$, and a $20\Delta x$ wave with $A = 1$. The blue line represents the sum of the four waves; it can be viewed as a 'full' wave. The atmosphere is comprised of many 'full' waves, each of which can be viewed as the sum of many waves of varying amplitude; here, we consider only four contributing waves for simplicity.

Let us assume that the 'full' wave is advected eastward at a constant velocity U. The true solution for advection is simply the translation of this wave eastward with no change in its structure.

As you may recall, 1-D advection can be written as:

$$\frac{\partial h}{\partial t} = -U\frac{\partial h}{\partial x}$$

When we approximate the partial derivatives with finite difference approximations derived from truncated Taylor series approximations, the numerical representation of the ‘full’ wave will depart from the true solution. These departures, generically associated with truncation error, manifest in this case as implicit numerical damping (for some differencing schemes) and numerical dispersion.

Consider the *damping only* case. If dispersion is not acting, then the wave is non-dispersive and it and its energy travel at a constant value that is equal for all wavelengths. Thus, the ‘full’ wave and all of its components will be advected eastward at a constant velocity. If the damping operator is not wavelength-selective (i.e., it dampens all wavelengths equally), then the amplitude of the ‘full’ wave and its components will dampen uniformly with time. If the damping operator is wavelength-selective (i.e., it dampens shorter wavelengths more than longer wavelengths), then the amplitudes of the individual waves will dampen non-uniformly with time: the shorter waves will dampen more than the longer waves. This will result in the ‘full’ wave more closely resembling the $20\Delta x$ wave with time, but the wave and all of its components will otherwise stick together.

Consider the *dispersion only* case. Dispersion results in waves of different wavelengths moving at different velocities; in most cases, dispersion results in the shortest wavelengths moving relatively slow and the longest wavelengths moving at approximately the correct velocity. In the absence of damping/diffusion, dispersion will result in the individual components to the ‘full’ wave moving at different speeds with time – in other words, they will spread apart. This will change the structure of the ‘full’ wave as the sum of the individual waves changes at each grid point. This can manifest as an *apparent* change in amplitude even though *the amplitudes of each wave are not changing in the absence of damping*!

In the *damping plus dispersion* case, both damping and dispersion act: the individual waves spread apart *and* their amplitudes are dampened (whether in a wavelength-selective manner or not).

To isolate the effects of implicit damping and dispersion would require using Fourier analysis to decompose the ‘full’ wave into its individual components over a wide range of wavelengths and then examining how the amplitudes of those individual components evolve in time. Although this is possible, it is beyond the scope of what we intend to cover in this class.

For this class, it is more important to recognize that damping and dispersion are possible with odd-order-accurate differencing schemes, but implicit damping does not occur for even-order-accurate differencing schemes. Both damping/diffusion and dispersion can impact solution quality, each to varying extents depending upon the specific spatial and temporal differencing schemes chosen: a greater impact for less-accurate schemes that truncate more terms from the Taylor series expansion and a smaller impact for more-accurate schemes that truncate fewer such terms.