Introduction to Spectral Modeling

Both spectral and finite-element methods for numerical weather prediction are applications of the Galerkin method, which states that the error in approximating some function $f(x)$ by a sum of basis functions is orthogonal to the basis functions themselves. In layman’s terms, this means that the “approximation” is actually exact and not an approximation at all.

The general form of approximating some function $f(x)$ by a sum – here, infinite – of basis functions is given by:

$$f(x) = \sum_{n=1}^{\infty} f_n \theta_n(x)$$

Here, $n$ is wavenumber, which is inversely proportional to the wavelength. The $f_n$ are coefficients that are not a function of $x$; these modulate the amplitude of the basis functions. The $\theta_n(x)$ is the basis function. The specific value of the coefficient $f_n$, and the specific form of the basis function $\theta_n(x)$, depends upon wavenumber $n$.

If the function $f$ is a function of both time $t$ and space $x$, the above equation takes the form:

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t) \theta_n(x)$$

Here, the spatial variability is captured by the basis functions. Temporal variability is captured by the coefficients on the basis functions.

For spectral methods, the basis functions are sometimes referred to as spherical harmonics. In these models, the basis functions take the form of Fourier series, where zonal variability in the dependent variables is represented by series of sine and cosine waves of varying wavelength, and Legendre functions, which depict meridional variability in the dependent variables. These basis functions are continuous across the sphere. Finite-element methods use basis functions of similar yet discontinuous formulation.

This highlights the inherent motivation of spectral methods for numerical weather prediction: to represent spatial – here, $x$/longitude and $y$/latitude – variability in the dependent variables analytically. This allows for exact solutions to be obtained for the partial derivatives in the $x$/longitude and $y$/latitude directions. By contrast, grid-point models use finite difference approximations to calculate these partial derivatives. Thus, truncation error associated with finite difference approximations in the spatial directions is not a factor with spectral methods.
However, it should be noted that finite difference approximations are used with spectral methods to advance the model forward in time. They are also often used to handle partial derivatives in the vertical direction. Thus, truncation error is not entirely eliminated. Further, as we cannot represent an infinite number of waves within spectral representations of the dependent variables, we must truncate the above series at a maximum wavenumber or, equivalently, minimum wavelength. Smaller-scale variability is lost in doing so. Finally, certain terms in the primitive equations cannot be handled with spectral methods. Thus, spectral methods are not a panacea for all numerical approximations associated with numerical weather prediction.

Spectral methods are most commonly used for global numerical weather prediction, with the primitive equations cast in spherical rather than Cartesian coordinates, at operational forecast centers. Both the NCEP Global Forecast System and the ECMWF Integrated Forecast System models are spectral models, for instance. However, global models need not be spectral models, nor is the use of spectral models limited to global applications. The course textbook describes several applications of spectral models for regional weather prediction, although it should be noted that most limited-area or regional models are grid-point in formulation.

Zonal Variability: Fourier Series

Fourier series, with sines and cosines serving as the basis functions, are used to represent zonal variability in the dependent variables within spectral models. For a generic function $A(x)$, its Fourier series expansion is given by:

$$A(x) = \sum_{n=0}^{\infty} [a_n \cos(nx) + b_n \sin(nx)]$$

Here, $n$ is the zonal wavenumber and is an integer, $k$ is equal to $2\pi/L$, $L$ is the domain length, and $a_n$ and $b_n$ are real-valued Fourier coefficients. If $A$ were to also be a function of time $t$, then the Fourier coefficients would also be a function of $t$ but the basis functions themselves would not.

The above expansion can instead be represented in terms of exponential functions via use of Euler’s relations, where:

$$e^{inx} = \cos(nx) + i\sin(nx)$$

$$e^{-inx} = \cos(nx) - i\sin(nx)$$

If we add these two expressions together, we obtain:

$$e^{inx} + e^{-inx} = 2\cos(nx)$$

Likewise, if we subtract the second expression from the first, we obtain:
\[ e^{inkx} - e^{-inkx} = 2i \sin(nkx) \]

If we substitute these two relations into the Fourier series expansion, we obtain:

\[ A(x) = \sum_{n=0}^{\infty} \left[ \frac{a_n}{2} \left( e^{inkx} + e^{-inkx} \right) + \frac{b_n}{2i} \left( e^{inkx} - e^{-inkx} \right) \right] \]

Grouping like exponential terms, this can be rewritten as:

\[ A(x) = \sum_{n=0}^{\infty} \left[ e^{inkx} \left( \frac{a_n}{2} + \frac{b_n}{2i} \right) + e^{-inkx} \left( \frac{a_n}{2} - \frac{b_n}{2i} \right) \right] \]

If we consider the case where \( n = 0 \), then:

\[ A(x)_{n=0} = \left( \frac{a_0}{2} + \frac{b_0}{2i} \right) + \left( \frac{a_0}{2} - \frac{b_0}{2i} \right) = a_0 \]

Thus,

\[ A(x) = a_0 + \sum_{n=1}^{\infty} \left[ e^{inkx} \left( \frac{a_n}{2} + \frac{b_n}{2i} \right) + e^{-inkx} \left( \frac{a_n}{2} - \frac{b_n}{2i} \right) \right] \]

For \( i = \sqrt{-1} \), if one divides both sides by \( i \), then \( 1 = \frac{\sqrt{-1}}{i} \). Multiplying both sides by \( \sqrt{-1} \) results in \( \sqrt{-1} = \frac{-1}{i} \), or \( i = -\frac{1}{i} \). Using this, the combinations of Fourier coefficients above can be simplified as follows:

\[
\left( \frac{a_n}{2} + \frac{b_n}{2i} \right) = \frac{2ia_n + 2b_n}{4i} = \frac{a_n + \frac{1}{i}b_n}{2} = \frac{a_n - ib_n}{2} \equiv c_n
\]

\[
\left( \frac{a_n}{2} - \frac{b_n}{2i} \right) = \frac{2ia_n - 2b_n}{4i} = \frac{a_n - \frac{1}{i}b_n}{2} = \frac{a_n + ib_n}{2} \equiv c_{-n}
\]

Thus, we obtain:

\[ A(x) = a_0 + \sum_{n=1}^{\infty} \left[ c_n e^{inkx} + c_{-n} e^{-inkx} \right] = a_0 + \sum_{n=1}^{\infty} \left[ c_n e^{inkx} \right] + \sum_{n=1}^{\infty} \left[ c_{-n} e^{-inkx} \right] \]
The last summation above can equivalently be written as 
\[ \sum_{n=-\infty}^{-1} c_n e^{inkx} \], such that:

\[ A(x) = a_0 + \sum_{n=1}^{\infty} c_n e^{inkx} + \sum_{n=-\infty}^{-1} c_n e^{inkx} = \sum_{n=-\infty}^{\infty} c_n e^{inkx} \]

where the \( a_0 \) has been re-encapsulated within the exponential. In other words, \( A(x) \) has been expressed as the sum of an infinite number of waves with amplitude \( c_n \) and structure \( e^{inkx} \).

The spatial variability in \( A(x) \) can be determined analytically:

\[ \frac{\partial A}{\partial x} = \sum_{n=-\infty}^{\infty} [nke^{inkx}] = \sum_{n=-\infty}^{\infty} [nke^{inkx}] \]

In other words, no finite difference approximations are necessary to quantify the variability in \( A \) over the zonal, or \( x \), direction!

An Introduction to Model Resolution

As we cannot represent an infinite series, the one-dimensional series is truncated with respect to a maximum allowable zonal wavenumber \( n = N \), i.e.,

\[ A(x) = \sum_{n=-N}^{N} [c_n e^{inkx}] \]

The chosen value of \( N \), given the inverse relationship between wavenumber and wavelength, determines the equivalent horizontal grid increment for the spectral model in the zonal direction. Note that as with grid-point models, this is not equivalent to the horizontal resolution, even though these terms are often used interchangeably. As spectral models use fast Fourier transforms to efficiently evaluate forward and inverse Fourier transforms, \( N \) should be a product of small prime numbers (e.g., \( N = 2^a 3^b 5^c \), where \( a, b, \) and \( c \) are all generic exponents).

Though there exist different interpretations of the relationship between \( N \) and the equivalent horizontal grid increment, the most commonly-used relationship is given by:

\[ L = \frac{\pi a}{N} \]

where \( a \) is the radius of the Earth, approximately equal to 6371 km. For example, for \( N = 799 \), \( L = 25.05 \) km. Formally, this \( L \) is the equivalent horizontal grid increment in the zonal direction at the Equator. It is typically used for models that utilize semi-Lagrangian spatial differencing and triangular truncation. Currently, the GFS operational deterministic model utilizes \( N = 1534 \), such
that $L = 13.05$ km. Likewise, the ECMWF operational deterministic model utilizes $N = 1279$, such that $L = 15.65$ km. The effective resolution of the ECMWF model is approximately $8L$.

A Simplistic Meteorological Application in One Dimension

Consider temperature $T$. Let us assume for simplicity that $T$ is only a function of $x$. We can represent $T(x)$ as the sum of a finite series:

$$T(x) = \sum_{n=-N}^{N} c_n e^{i n \pi x / L}$$

The Fourier coefficients $c_n$ can be obtained via a forward Fourier transform, which can be approximated by the following summation using what is known as trapezoidal quadrature:

$$c_n = \frac{1}{L} \sum_{j=1}^{J} T(x_j) e^{-i n \pi x_j / L}$$

Here, $L$ again represents the domain length, $j$ is a grid point index in the zonal direction, and $J$ is the number of regularly-spaced grid points in the $x$-direction. This summation exactly equals the integral form of the forward Fourier transform when $J$ is greater than or equal to $3N + 1$. This criterion also ensures that aliasing, or the erroneous redistribution of energy (as represented by amplitude) between wavenumbers, does not contaminate the solutions.

Given knowledge of $T(x)$ at all regularly-spaced grid points $j$, one can obtain the Fourier coefficients $c_n$ for the spectral representation of $T(x)$. $T(x)$ can subsequently be retrieved via the inverse Fourier transform, which takes the form of the equation for $T(x)$ above.

Meridional Variability: Legendre Functions

Legendre functions serve as the basis functions to represent meridional variability in the dependent variables within spectral models. For a generic function $A(\lambda, \phi)$, where $\lambda$ is longitude and $\phi$ is latitude, its spectral expansion is given by:

$$A(\lambda, \phi) = \sum_{n=-N}^{N} \sum_{m=|n|}^{M(n)} \alpha_{mn}^n P_m^n(\mu) e^{i n \lambda}$$

Here, $n$ is the zonal wavenumber, $N$ is the highest-permitted zonal wavenumber, $m$ is the order of the associated Legendre function (where $m$ is a positive integer), $M$ is the highest-permitted order of the associated Legendre function and is a function of $n$, $\alpha_{mn}^n$ are the spectral coefficients, and $P_m^n(\mu)$ are the Legendre polynomial functions and are a function of $\mu = \sin \phi$. 

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One may equivalently write:

\[ A(\lambda, \phi) = \sum_{n=-N}^{N} \sum_{m=-M}^{M} \alpha_m^n Y_m^n(\lambda, \phi) \]

where \( Y_m^n(\lambda, \phi) = P_m^n(\mu) e^{im\lambda} \).

\( Y_m^n \) are the spherical harmonics and form the basis to solutions to Laplace’s equation (e.g., \( \nabla^2 f = 0 \) for any generic variable \( f \)) on the sphere. That equations akin to Laplace’s equation describe atmospheric wave propagation is the fundamental basis for the utilization of Fourier series and Legendre polynomials to represent spatial variability within the atmosphere.

The Legendre polynomials can be obtained using Rodrigues’ formula, where:

\[ P_m^n(\mu) = \frac{(1-\mu^2)^{n/2}}{2^m m!} \frac{d^{m+n}}{d\mu^{m+n}}(\mu^2 - 1)^m \quad \text{for } |\mu| \leq 1 \]

The \( d \) terms reflect derivatives of order \( m+n \), and \( \mu, m, \) and \( n \) are all as defined above. Note that \( P_m^n(\mu) = 0 \) for \( m < n \). In other words, the order \( m \) of the Legendre polynomial must be greater than or equal to the wavenumber \( n \) at each wavenumber. This is because of the derivative term in the equation above; where \( m < n \), the highest power on \( \mu \) will be less than that of the derivative, such that the derivative itself – and thus \( P_m^n(\mu) \) – is zero. This is why the second summation in the expansion of \( A(\lambda,\phi) \) above starts at \( m = |n| \).

Legendre polynomials and their derivatives for varying \( m \) and \( n \) are related to one another by what are known as recurrence relations. For large \( N \) and \( M \), the use of recurrence relations simplifies the otherwise arduous task of defining all \( P_m^n \) from the expression above. The two most commonly-used recurrence relations are given below:

\[ \mu P_m^n = \varepsilon_m^n P_{m-1}^n + \varepsilon_{m+1}^n P_{m+1}^n, \quad \text{where } \varepsilon_m^n = \sqrt{\frac{m^2 - n^2}{4m^2 - 1}} \]

\[ (1-\mu^2) \frac{dP_m^n}{d\mu} = (m+1)\varepsilon_m^n P_{m-1}^n - m\varepsilon_{m+1}^n P_{m+1}^n \]

The first relates a Legendre polynomial \( P_m^n \) to those of adjacent orders \( m-1 \) and \( m+1 \). The second works similarly, except for its first derivative.

\[ A \text{ Simplicitic Meteorological Application in Two Dimensions} \]
Consider again temperature $T$. Let us assume that $T$ is a function of both latitude $\phi$ (where $\mu = \sin \phi$) and longitude $\lambda$. The spectral expansion of $T(\lambda, \mu)$ is given by:

$$T(\lambda, \mu) = \sum_{n=-N}^{N} \sum_{m=|n|}^{M(n)} \hat{T}_m^n P_m^n(\mu)e^{in\lambda}$$

The spectral coefficients $\hat{T}_m^n$ can be obtained first via a forward Fourier transform:

$$\hat{T}_m^n(\mu) = \frac{1}{L} \sum_{j=1}^{J} T(\lambda_j, \mu) e^{-in\lambda_j}$$

Followed by a Legendre transform of the form:

$$\hat{T}_m^n = \sum_{k=1}^{K} W(\mu_k) \hat{T}_m^n(\mu_k) P_m^n(\mu_k)$$

Here, $W(\mu_k)$ are Gaussian weights, $\mu_k$ are the Gaussian latitudes, $k$ is a grid-point index in the meridional direction, and $K$ is the number of grid points in the meridional direction. We will discuss the Gaussian weights and Gaussian latitudes in more detail shortly.

The summation forms of the Fourier and Legendre transformations are approximations to their exact integral forms. Again, the summation equals the integral form of the Fourier transform when $J$ is greater than or equal to $3N + 1$. The summation equals the integral form of the Legendre transform when $K$ is greater than or equal to $(3N + 1)/2$ for triangular truncation and greater than or equal to $5N/2$ for rhomboidal truncation. As before, these criterion also ensure that the solutions are unaffected by aliasing.

Given knowledge of $T(\lambda, \mu)$ at all regularly-spaced grid points $j$ and $k$, one can obtain the spectral coefficients $\hat{T}_m^n$ for the spectral representation of $T(\lambda, \mu)$. $T(\lambda, \mu)$ can be retrieved via inverse Legendre and inverse Fourier transforms, which combined take the form of the equation for $T(\lambda, \mu)$ above.

**Truncation in Two Dimensions**

In the above, we demonstrated that the order $m$ of the Legendre polynomials – akin to the meridional wavenumber – is a function of the zonal wavenumber $n$. Specifically, we stated that $m$ must be greater than or equal to $n$ for all $n$. However, what determines the highest-permitted value of $m$, which we previously denoted as $M(n)$ and which is related to the truncation in the meridional direction? There are two methods by which this value of $M$ can be specified:
• **Triangular truncation**: $M$ is equal to $N$ (the highest-permitted zonal wavenumber) at all $n$. In other words, an identical maximum wavenumber is allowed in both the zonal and meridional directions, with fewer allowable $m$ as $|n|$ grows increasingly large because $m$ must be greater than or equal to $n$. This truncation allows for uniform model resolution across the sphere in both the zonal and meridional directions. The allowable $m$ as a function of $n$ for triangular truncation is depicted in Figure 1 (left).

• **Rhomboidal truncation**: $M$ is equal to a constant value for all $n$, where $|n| + N = M$. Thus, the allowable number of individual $m$ is equal to $N$. This truncation allows for uniform model resolution across the sphere in the zonal direction. Model resolution across the sphere in the meridional direction is variable and is highest near the poles. The allowable $m$ as a function of $n$ for rhomboidal truncation is depicted in Figure 1 (right).

![Figure 1](image.png)

**Figure 1.** Schematics showing the relationship between the highest-allowed order $M$ of Legendre polynomial, zonal wavenumber $n$, and highest-allowed zonal wavenumber $N$ for (left) triangular truncation and (right) rhomboidal truncation. Adapted from Warner (2011), their Figure 3.17.

Most modern spectral models utilize triangular truncation. It offers uniform resolution across the sphere in both the meridional and zonal directions. And, where each cell in Figure 1 can be taken to represent a single $P_m^n$, there are fewer Legendre polynomials that need to be calculated when using triangular truncation as compared to rhomboidal truncation. Rhomboidal truncation does offer the benefit of having less variance at short wavelengths than does triangular truncation.

**The Polar Problem**

The horizontal wind components $u$ and $v$ pose a singularity at the poles. At the North Pole, for instance, every direction is south. At the South Pole, every direction is north. This poses a
challenge for representing $u$ and $v$ via the spectral transform method near the poles. Instead, Robert functions are used to recast $u$ and $v$ as follows:

$$U = \frac{u \cos \theta}{a} \quad V = \frac{v \cos \theta}{a}$$

where $a$ is the radius of the Earth and $\theta$ is latitude, expressed in radians, such that $\theta = \pi/2$ at the North Pole and $\theta = -\pi/2$ at the South Pole. As a result, $U = V = 0$ at the poles, eliminating the singularity. Elsewhere, $U$ and $V$ scale with $u$ and $v$ as a function of latitude. Using $U$ and $V$ does not change the direction of the wind, nor does it change the relative magnitude of $u$ versus that of $v$, given that both $u$ and $v$ are scaled by the same factor.

**Reduced Gaussian Grids**

Earlier, we defined $J$ and $K$ as the number of grid points in the zonal and meridional directions, respectively, on which a dependent variable – such as temperature – is defined in physical space. The chosen $J$ and $K$ define what is known as the transformation grid. The need for such a grid can be illustrated in light of input to and output from the model:

- Data used to initialize numerical models, reflecting the optimal combination of a “first guess” from a previous model forecast and available observations, are available in physical space and not in spectral space. Having these data available on a uniform grid allows for the computation of the spectral coefficients at the initial time via forward Fourier (zonal) and Legendre (meridional) transforms.

- Our interpretation of meteorological fields is tied to their spatially-dependent structure, not to the wavenumber-dependent structure of their spectral coefficients. In other words, we think in terms of the dependent variables – $u$, $v$, $p$, etc. – in space and not their spectral coefficients in wavenumber space. Thus, we need a grid onto which we transform variables from spectral space to physical space to facilitate the analysis and interpretation of meteorological fields.

There also exist numerical considerations underlying the need for a transformation grid:

- Non-linear terms, or those involving the product of variables such as $u$ and $\frac{\partial u}{\partial x}$, is computationally intensive, adding significant run-time to spectral models if not addressed. This is typically addressed by first obtaining each term in spectral space, inverse transforming each to physical space and computing their product there, then transforming the product back to spectral space.
Forcing terms that are discontinuous, discrete, or localized in nature cannot be accurately represented by spherical harmonics and thus inherently cannot be represented by spectral methods. Many parameterized processes are discontinuous or discrete. These, too, are typically handled in physical space.

For spectral models, the transformation grid takes the form of a reduced Gaussian grid. As the horizontal coordinates of spectral models are latitude $\phi$ and longitude $\lambda$, this grid is a latitude-longitude grid. Meridians, or lines of constant longitude, converge at the poles, which results in the physical distance between points on this grid in the zonal direction becoming infinitesimally small near the poles. To avoid the computational issues that this poses, a reduced form of this latitude-longitude grid is utilized, wherein the number of zonal grid points is reduced as latitude increases. It has been shown that the use of a reduced grid does not compromise the solution relative to that when a regular grid is used.

The characterization of this grid as Gaussian arises from the definition of the latitudes of the meridional grid points $k$. The chosen latitudes, known as Gaussian latitudes, satisfy the relationship $P_k^0(\mu) = 0$, where $\mu = \sin \phi$ and $P$ are the Legendre polynomials. Substituting these relationships into Rodrigues’ formula, one obtains:

$$P_k^0(\mu) = \frac{1}{2^k K!} \frac{d^k}{d\mu^k} (\mu^2 - 1)^k = 0$$

From this equation, the roots (allowable solutions) of $\mu$ are found, which then define the latitudes of the reduced Gaussian grid. There exists a Gaussian weight $W$ at each meridional grid point $k$, defined as:

$$W_k(\mu_k) = \frac{2(1 - \mu_k^2)}{KP_{K-1}^0(\mu_k)}$$

Here, $\mu_k$ refers to a Gaussian latitude, and $P_{K-1}^0$ are taken to be a function of $\mu_k$ and not multiplied by $\mu_k$. These Gaussian weights enter into the Legendre transform of a Fourier-transformed field so as to obtain the spectral coefficients for that field.

Spectral Modeling in Practice

Having introduced the fundamental concepts associated with spectral modeling, we now wish to illustrate how spectral modeling works in practice. Note that the spectral model equations themselves are cast in terms of the spectral coefficients using the definitions presented earlier in this lecture. For a given model cycle, we begin with model dependent variables on the reduced Gaussian grid, as defined above. The spectral coefficients for each variable at the initial time are
obtained using forward Fourier and Legendre transforms. Partial derivatives in the spatial dimension are computed analytically utilizing the transformed variables.

Next, all variables needed to compute non-linear forcing terms, parameterized tendencies, and possibly vertical derivatives are transformed from spectral space to physical space using an inverse Legendre transform followed by an inverse Fourier transform. These terms are then computed using conventional means in physical space. Advection terms, which are one type of non-linear forcing term, may be computed in physical space using flow-following or grid-fixed methods. The resulting fields are then transformed back to spectral space using forward Fourier and Legendre transforms. With all forcings represented in spectral space in terms of the spectral coefficients, the spectral coefficients at the next time step may be obtained using conventional time differencing methods.

At selected output times and at the end of the model run, inverse Legendre and Fourier transforms are used to transform from spectral space to physical space, at which time the output can be visualized and interpreted using conventional means.

**Finite Volume Methods**

Grid-point models represent the value of model dependent variables at specified grid points in three dimensions, which can be taken to be the center of a grid cell (box, triangle, hexagon, etc.) or somewhere along its perimeter. These values are assumed to be representative of the values of the model dependent variables over the grid increment. It is also possible for grid-point models to instead consider the grid-cell-integrated values of model dependent variables. Models that do so are considered to be finite-volume models.

Finite-volume models take advantage of the flux form of the primitive equations, wherein advection and divergence terms are combined into a single flux divergence term. The divergence theorem states that the volume integral of the divergence of a vector is equal to the surface integral of the normal \( n \) component of the vector. Applied to the flux divergence term in the continuity equation \( \nabla \cdot (\mathbf{v\rho}) \),

\[
\iiint_v \nabla \cdot (\mathbf{v\rho}) dV = \iint_S (\mathbf{v\rho}) \cdot \mathbf{n} dS
\]

In other words, the volume integral of the flux divergence of density is equal to the area integral of the flux of density into or out of the grid cell.

Or, cast in terms of the full continuity equation and dividing by the volume \( V \):

\[
\frac{1}{V} \iiint_v \frac{\partial \rho}{\partial t} dV = -\frac{1}{V} \iint_S (\mathbf{v\rho}) \cdot \mathbf{n} dS
\]
where the term on the left-hand side of this equation can be written as:

\[
\frac{1}{V} \iiint_V \frac{\partial \rho}{\partial t} dV = \frac{\partial}{\partial t} \left( \frac{1}{V} \iiint_V \rho dV \right) = \frac{\partial \bar{\rho}}{\partial t}
\]

Since the volume is constant with time, the time derivative can be pulled out of the integral. The resulting term is simply the volume-averaged density – volume-integrated density divided by the volume. The time rate of change of this volume-averaged density is thus simply equal to the flux of density into or out of the grid cell.

For cases where model dependent variables vary linearly in all three spatial dimensions, finite difference and finite volume methods are numerical similar and result in similar forecasts. However, for cases where model dependent variables do not vary linearly between grid points, finite-volume methods produce superior results.

Finite-volume models are conservative in nature for the model dependent variables that are represented in cell-integrated form. For instance, mass conservation can be achieved by applying the finite-volume method to the continuity equation (and thus to density). Other properties such as total energy, angular momentum, and entropy can be conserved if the finite-volume method is applied appropriate model dependent variable(s) and their respective prognostic equation(s). The NOAA Finite-volume Icosahedral Model, or FIM, and NCAR/LANL Model for Prediction Across Scales, or MPAS, are both examples of modern finite-volume models, while the NCAR Community Atmosphere Model can be run in a finite-volume configuration.

Note that finite-volume models often, though do not exclusively, utilize semi-Lagrangian spatial differencing, which evaluate spatial variability in model dependent variables using hybrid flow-following rather than grid-fixed (or Eulerian) techniques. We will discuss semi-Lagrangian spatial differencing methods in greater detail in a subsequent lecture.