Linear Numerical Stability

Introductory Remarks

In the computational sense, **stability** is defined by the temporal evolution of the model solution. Does the model solution grow exponentially with time, leading to floating point overflow – one or more model variables becoming too large for the computer to represent – and the model crashing? If it does, the model is said to be computationally unstable. In general, we assess stability in the context of identifying the conditions under which this occurs.

The *CFL criterion*, in its most general form, is a stability criterion for a linear advection term: under what conditions (U, Δt, and Δx) does the model solution become unstable? In this lecture, we will formally derive the CFL criterion for two types of terms, (linear) advection and explicit numerical diffusion – for several combinations of temporal and spatial finite differencing schemes. We will find that the CFL criterion typically varies as a function of the differencing schemes used.

As in the atmosphere, such as for vertical parcel displacements, there are three types of numerical stability that may exist, listed roughly from least to most common:

- **Absolutely unstable**: no matter what values are chosen for the dependent parameters (e.g., U, Δt, and Δx), the model will always crash.
- **Absolutely stable**: no matter what values are chosen for the dependent parameters, the model will never crash due to floating point overflow.
- **Conditionally stable**: so long as the chosen values for the dependent parameters adhere to an appropriate stability criterion, the model solution will remain stable.

All terms in the primitive equations contribute to the numerical stability of the model solution. Most problematic are the advection terms, whether they are linear (e.g., the product of a velocity and a partial derivative of a mass-related field) or non-linear (e.g., the product of a velocity and the partial derivative of that velocity) in nature. Advection terms, and in particular linear advection terms, form the basis for our investigation in this lecture.

Consider a one-dimensional advection equation for a generic variable h that is advected by a mean velocity U:

$$\left.\frac{\partial h}{\partial t}\right|_j^\tau = -U\left.\frac{\partial h}{\partial x}\right|_j^\tau$$

Here, subscripts indicate that the terms are evaluated at a point j along the x-axis, while superscripts indicate that the terms are evaluated at a time step τ.

We wish to specify harmonic, or wave-like, solutions for h of the form:

$$h = \hat{h}e^{i(kx-\omega t)}$$

Here, $\hat{h}$ is amplitude, k is a zonal wavenumber equal to $2\pi/L$, L is wavelength, and ω is frequency (s^{-1}) and equal to Uk. In the above, the exponential function specifies wave-like structure through Euler's formula, where $e^{i\theta} = \cos\theta + i\sin\theta$.

Before proceeding, it is helpful for us to note that any feature on a model grid is truly represented by the summation of an infinite number of waves, with wavenumber k from 1 (one wavelength on the model grid) to infinity (an infinite number of waves, with infinitesimally small wavelength, on the model grid), of varying amplitude. Keep this in mind as we consider wavenumber dependence in the linear stability criteria we will soon derive, as some differencing schemes will be stable for some wavelengths and unstable for others. As we will see in our next lecture, this is also important for the artificial dispersion of waves that can result from the chosen differencing scheme.

We assume that frequency ω has both real and imaginary components, such that $\omega = \omega_R + i\omega_I$. Though ω_I itself is real-valued, the leading i makes it imaginary. If we substitute this definition for ω into the definition for h, we obtain:

$$h = \hat{h}e^{i(kx-\omega t)} = \hat{h}e^{i(kx-(\omega_R+i\omega_I)t)} = \hat{h}e^{\omega_I t}e^{i(kx-\omega_R t)}$$

It is clear that the amplitude of h is no longer equal to a constant value $\hat{h}$ but is now a function of time, through $e^{\omega_I t}$, such that ω_I determines whether the amplitude of h **grows exponentially** ($\left|e^{\omega_I t}\right| > 1$, such that $\omega_I > 0$), **remains constant** ($\left|e^{\omega_I t}\right| = 1$, such that $\omega_I = 0$), or **dampens** with time ($\left|e^{\omega_I t}\right| < 1$, such that $\omega_I < 0$). Consequently, to determine linear numerical stability, we need to determine the conditions under which $\left|e^{\omega_I t}\right| \le 1$ (for stability) and $\left|e^{\omega_I t}\right| > 1$ (for unstable solutions).

Linear Stability of Forward-in-time, Backward-in-space Finite Differences

Let us examine the stability of the forward-in-time, backward-in-space combination of finite difference schemes. Though this is a combination of differencing schemes that is associated with particularly large truncation error, it also provides a fairly direct evaluation of numerical stability. In this case, the one-dimensional advection equation becomes:

$$\frac{h_j^{\tau+1} - h_j^{\tau}}{\Delta t} = -U\frac{h_j^{\tau} - h_{j-1}^{\tau}}{\Delta x}$$

Or, equivalently:

$$h_j^{\tau+1} - h_j^{\tau} = -\frac{U\Delta t}{\Delta x}\left(h_j^{\tau} - h_{j-1}^{\tau}\right)$$

Note that $x = j\Delta x$, such that the location is equal to the grid point j multiplied by the grid spacing Δx, and $t = \tau\Delta t$, such that the time is equal to the time step # τ multiplied by the time step Δt. Given map projections, the Δx in the above is that on the Earth (Δx_e), which is often smaller than that on the model grid (Δx_g).

The wave-like solution for h can be rewritten in terms of j and τ:

$$h = \hat{h}e^{\omega_I t}e^{i(kx-\omega_R t)} = \hat{h}e^{\omega_I \tau\Delta t}e^{i(kj\Delta x-\omega_R \tau\Delta t)}$$

If we substitute this solution into the finite difference form of the 1-D equation above, we obtain:

$$\hat{h}e^{\omega_I(\tau+1)\Delta t}e^{i(kj\Delta x-\omega_R(\tau+1)\Delta t)} - \hat{h}e^{\omega_I \tau\Delta t}e^{i(kj\Delta x-\omega_R \tau\Delta t)} = -\frac{U\Delta t}{\Delta x}\left(\hat{h}e^{\omega_I \tau\Delta t}e^{i(kj\Delta x-\omega_R \tau\Delta t)} - \hat{h}e^{\omega_I \tau\Delta t}e^{i(k(j-1)\Delta x-\omega_R \tau\Delta t)}\right)$$

Divide by a common factor of $\hat{h}e^{\omega_I \tau\Delta t}e^{i(kj\Delta x-\omega_R \tau\Delta t)}$ to obtain:

$$e^{\omega_I \Delta t}e^{-i\omega_R \Delta t} - 1 = -\frac{U\Delta t}{\Delta x}\left(1 - e^{-ik\Delta x}\right)$$

Note the difference in this equation from that which is equation (3.38) in the course text; here, there is a leading negative sign in the last exponential, which is correct, whereas the course text lacks such a leading negative despite obtaining the correct solution at the end.

The exponentials involving i can be rewritten using Euler's formula, where $e^{i\theta} = \cos\theta + i\sin\theta$ and $e^{-i\theta} = \cos\theta - i\sin\theta$. Doing so, we obtain:

$$e^{\omega_I \Delta t}\left(\cos(\omega_R \Delta t) - i\sin(\omega_R \Delta t)\right) - 1 = -\frac{U\Delta t}{\Delta x}\left(1 - \left(\cos(k\Delta x) - i\sin(k\Delta x)\right)\right)$$

If we separate this equation into its real (top) and imaginary (bottom) parts, we obtain:

$$e^{\omega_I \Delta t}\cos(\omega_R \Delta t) - 1 = -\frac{U\Delta t}{\Delta x}\left(1 - \cos(k\Delta x)\right)$$

$$-ie^{\omega_I \Delta t}\sin(\omega_R \Delta t) = -i\frac{U\Delta t}{\Delta x}\sin(k\Delta x)$$

Or, written equivalently,

$$e^{\omega_I \Delta t}\cos(\omega_R \Delta t) = 1 - \frac{U\Delta t}{\Delta x}\left(1 - \cos(k\Delta x)\right)$$

$$e^{\omega_I \Delta t}\sin(\omega_R \Delta t) = \frac{U\Delta t}{\Delta x}\sin(k\Delta x)$$

We wish to combine these equations so that we can obtain an equation for $e^{\omega_I \Delta t}$. To do so, we want to eliminate ω_R, which is associated with wave propagation and dispersion. This can be done by squaring each equation and adding them together, since $\cos^2\theta + \sin^2\theta = 1$. Doing so, we obtain:

$$e^{\omega_I \Delta t} e^{\omega_I \Delta t} = \left(1 - \frac{U\Delta t}{\Delta x}\left(1 - \cos(k\Delta x)\right)\right)^2 + \left(\frac{U\Delta t}{\Delta x}\sin(k\Delta x)\right)^2$$

Taking the square root of both sides of this equation, we obtain:

$$\left|e^{\omega_I \Delta t}\right| = \sqrt{\left(1 - \frac{U\Delta t}{\Delta x}\left(1 - \cos(k\Delta x)\right)\right)^2 + \left(\frac{U\Delta t}{\Delta x}\sin(k\Delta x)\right)^2}$$

Note that we have taken the absolute value of the left-hand side to keep only the positive root. We are less interested in the sign of $e^{\omega_I \Delta t}$ as we are in whether its magnitude is greater than 1.

We can expand everything under the radical as follows:

$$\left(\frac{U\Delta t}{\Delta x}\sin(k\Delta x)\right)^2 = \frac{U^2(\Delta t)^2}{(\Delta x)^2}\sin^2(k\Delta x)$$

$$\begin{aligned}\left(1 - \frac{U\Delta t}{\Delta x}\left(1 - \cos(k\Delta x)\right)\right)^2 &= 1 - 2\frac{U\Delta t}{\Delta x}\left(1 - \cos(k\Delta x)\right) + \frac{U^2(\Delta t)^2}{(\Delta x)^2}\left(1 - \cos(k\Delta x)\right)^2 \\ &= 1 - 2\frac{U\Delta t}{\Delta x} + 2\frac{U\Delta t}{\Delta x}\cos(k\Delta x) + \frac{U^2(\Delta t)^2}{(\Delta x)^2}\left(1 - 2\cos(k\Delta x) + \cos^2(k\Delta x)\right)\end{aligned}$$

Adding these two equations, substituting for the resulting $\frac{U^2(\Delta t)^2}{(\Delta x)^2}\left(\sin^2(k\Delta x) + \cos^2(k\Delta x)\right)$ term, and combining like terms, we obtain:

$$\begin{aligned}\left|e^{\omega_I \Delta t}\right| &= \sqrt{1 - 2\frac{U\Delta t}{\Delta x} + 2\frac{U\Delta t}{\Delta x}\cos(k\Delta x) + 2\frac{U^2(\Delta t)^2}{(\Delta x)^2} - 2\frac{U^2(\Delta t)^2}{(\Delta x)^2}\cos(k\Delta x)} \\ &= \sqrt{1 + 2\frac{U\Delta t}{\Delta x}\left(\cos(k\Delta x) - 1 + \frac{U\Delta t}{\Delta x} - \frac{U\Delta t}{\Delta x}\cos(k\Delta x)\right)}\end{aligned}$$

For $(a+b)(c+d) = ab + ad + bc + bd$, if $a = \cos(k\Delta x)$, $b = -1$, $c = 1$, and $d = -\frac{U\Delta t}{\Delta x}$, the terms in the parentheses underneath the radical can be simplified:

$$\left|e^{\omega_I \Delta t}\right| = \sqrt{1 + 2\frac{U\Delta t}{\Delta x}\left[\left(\cos(k\Delta x) - 1\right)\left(1 - \frac{U\Delta t}{\Delta x}\right)\right]}$$

Recall that the value of $e^{\omega_I t}$ determines whether the amplitude of *h* grows, decays, or remains constant in time. We previously defined $t = \tau\Delta t$, such that $e^{\omega_I t} = e^{\omega_I \tau\Delta t} = \left(e^{\omega_I \Delta t}\right)^{\tau}$. Thus, the value of $\left|e^{\omega_I \Delta t}\right|$ determines how the amplitude of *h* will change *over one time step*, which is then raised to the power of τ (i.e., this amplitude change grows exponentially from one time step to the next).

The stability criterion above is a function of both the Courant number and of $k\Delta x$, which for $k = 2\pi/L$ becomes $2\pi(\Delta x/L)$, and is thus a function of the ratio of the horizontal grid spacing to the wavelength.

Let us consider a simple case: $\frac{U\Delta t}{\Delta x} = 1$. In that case, everything under the radical collapses to 1, such that $\left|e^{\omega_I \Delta t}\right| = 1$. This is **numerically stable**, with *no change in amplitude* with time.

What about when $\frac{U\Delta t}{\Delta x} \neq 1$? Note that the largest-possible value of Δx is *L/2*, defining a grid of three points to represent the $2\Delta x$ wave. There, $k\Delta x = \pi$, with $\cos(\pi) = -1$. The smallest-possible value of Δx is approximately zero, defining a grid of an infinite number of points to resolve all waves. As $k\Delta x$ approaches zero, $\cos(k\Delta x)$ approaches 1. Thus, $\cos(k\Delta x)$ has allowable values between -1 and 1, such that $\cos(k\Delta x) - 1$ has allowable values between -2 and 0; in other words, it is always negative.

For $\frac{U\Delta t}{\Delta x} > 1$, $\left(1 - \frac{U\Delta t}{\Delta x}\right) < 0$. Thus, for $\cos(k\Delta x) - 1 < 0$, the number under the radical in the equation for $\left|e^{\omega_I \Delta t}\right|$ above is always greater than 1. Consequently, $\left|e^{\omega_I \Delta t}\right| > 1$, which defines a **numerically unstable** solution with *exponential amplitude growth* over time.

Conversely, for $\frac{U\Delta t}{\Delta x} < 1$, $\left(1 - \frac{U\Delta t}{\Delta x}\right) > 0$. Thus, for $\cos(k\Delta x) - 1 < 0$, the number under the radical in the equation for $\left|e^{\omega_I \Delta t}\right|$ above is always less than 1. Consequently, $\left|e^{\omega_I \Delta t}\right| < 1$, which defines a **numerically stable** solution with *exponential amplitude damping* over time.

Thus, for the backward-in-space, forward-in-time differencing scheme, the stability criterion is given by the generic form of the CFL criterion:

$$\frac{U\Delta t}{\Delta x} \leq 1$$

The precise degree to which the wave's amplitude grows or dampens over time is *a function of its wavelength* given the cos($k\Delta x$) term that appears in the equation for $\left|e^{\omega_I \Delta t}\right|$ above. Recall: $k = 2\pi/L$, such that $k\Delta x$ is equal to $2\pi\Delta x/L$. Thus, a wave of wavelength $2\Delta x$ will have $k\Delta x = \pi$ (with cos π = -1) while a wave of wavelength $20\Delta x$ will have $k\Delta x = \pi/10$ (with cos $\pi/10$ = 0.951). The precise degree to which the wave's amplitude grows or dampens over time is also a function of the Courant number itself, given the relationship to $\frac{U\Delta t}{\Delta x}$ in the expression for $\left|e^{\omega_I \Delta t}\right|$.

Both of these dependencies are illustrated in Figure 1, plotting the damping magnitude per time step for ten selected waves of wavelengths $2\Delta x$ to $50\Delta x$ over a range of stable Courant numbers. *For this specific combination of differencing schemes*, shorter wavelengths are damped to greater extent per time step than are longer wavelengths. Stated equivalently, as the horizontal grid spacing becomes small relative to the wavelength (e.g., more points to resolve the wave), the damping becomes smaller over the range of stable Courant numbers. There is no damping at Courant numbers of 0 or 1 with maximum damping for a Courant number of 0.5. Although not shown, exponential growth for Courant numbers greater than 1 is greatest for short wavelengths, just as is the damping for Courant numbers less than 1.

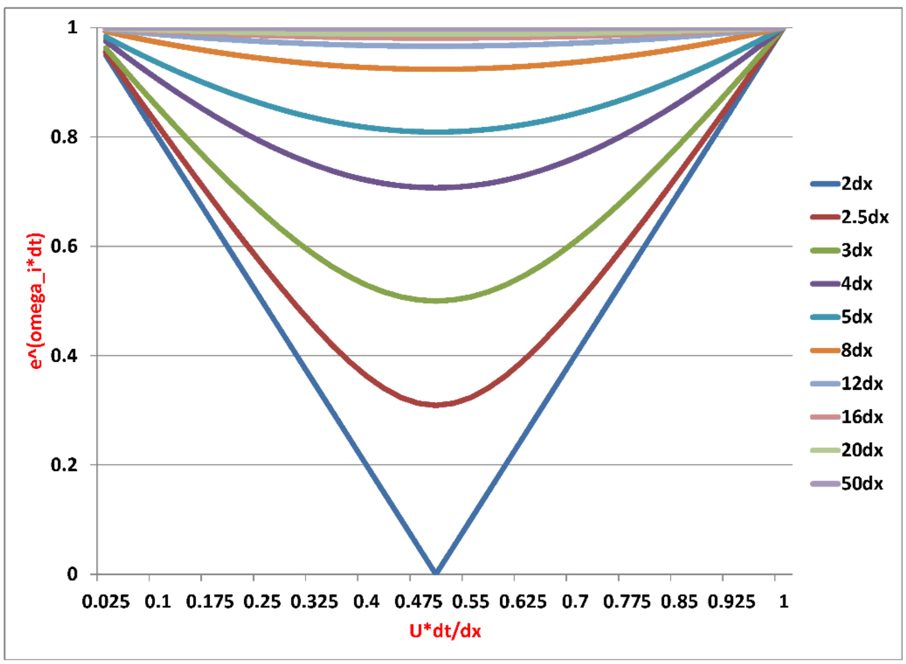

Figure 1. The value of $\left|e^{\omega_I \Delta t}\right|$ as a function of Courant number (numerically-stable values only) for waves of wavelength between $2\Delta x$ and $50\Delta x$. Please refer to the text for further details.

Linear Stability for Other Spatial and Temporal Differencing Schemes

The numerical stability of any combination of spatial and temporal differencing schemes can be assessed using the process outlined above. The course text describes this in some detail for the centered-in-time, centered-in-space scheme and states only the end results for the forward-in-time, second-order-accurate centered-in-space and centered-in-time, fourth-order-accurate centered-in-space differencing schemes. Here, we consider only basic insight for each; please refer to the course text, or consider conducting the derivations yourself, for more details.

(1) Forward-in-time, centered-in-space

$$\left|e^{\omega_I \Delta t}\right| = \sqrt{1+\left(\frac{U\Delta t}{\Delta x}\right)^2 \sin^2\left(k\Delta x\right)}$$

Both $\left(\frac{U\Delta t}{\Delta x}\right)^2$ and $\sin^2\left(k\Delta x\right)$ are positive-definite, such that the value under the radical is greater than 1 for all values of $k\Delta x$ and $\frac{U\Delta t}{\Delta x}$. Consequently, no matter the time step, this combination of differencing schemes is **numerically unstable**. As a result, this scheme is only used to advance the model for the first model time step when a centered-in-time scheme is used with the centered-in-space scheme, with the amplitude growth between the two time steps being acceptably small for this single instance.

(2) Centered-in-time (leapfrog), second-order-accurate centered-in-space

$$\frac{U\Delta t}{\Delta x}\sin\left(k\Delta x\right) \le 1$$

As was stated above, the allowable values of Δx range from $L/2$ to ~0, such that the allowable values of $k\Delta x$ range from π to ~0. The sin function in both cases evaluates to 0. Between 0 and π, the maximum value of $\sin(k\Delta x)$ is 1, which occurs when $k\Delta x = \pi/2$ (for $\Delta x = L/4$). This allows us to more generally state the stability criterion as:

$$\frac{U\Delta t}{\Delta x} \le 1$$

No matter the value of the sin function (between 0 and 1), so long as this criterion is met, numerical stability will be ensured.

(3) Centered-in-time (leapfrog), fourth-order-accurate centered-in-space

$$\frac{U\Delta t}{\Delta x} \le 0.73$$

Note that it can be shown that $\left|e^{\omega_I \Delta t}\right| = 1$ for all stable values of the Courant number for both (2) and (3). In other words, those schemes are either numerically stable *without damping* or they are numerically unstable, and this is true of all centered even-order-accurate differencing schemes. In contrast, all odd-order-accurate time and space differencing schemes are associated with implicit damping for stable values of the Courant number, with the specific damping magnitude dependent on wavelength and the Courant number.

The WRF-ARW Technical Document, as reproduced from Wicker and Skamarock (2002), lists the stability criteria for a wide range of spatial and temporal differencing schemes relative to the Courant number, where an X indicates that it is always numerically unstable:

	3rd Order	**4th Order**	**5th Order**	**6th Order**
Leapfrog	X	0.72	X	0.62
Runge-Kutta 2	0.88	X	0.30	X
Runge-Kutta 3	1.61	1.26	1.42	1.08

Note that these stability criteria are for one-dimensional linear advection, as we have considered to this point. We do not consider wavelength dependence in the above. Note that the default choices for WRF-ARW – Runge-Kutta 3 in time, 5th order in space – strike an effective balance between accuracy (higher-order in time and space) and computational efficiency (high Courant number) as compared to other available differencing schemes. Based upon this, the general guidance for the model time step Δt in WRF-ARW is 6*Δx, where Δx is input in km and the resulting Δt is in s.

As noted earlier, the *U* in the Courant number is not determined by the meteorology – and is instead determined by rapidly-moving sound and/or gravity waves – if these waves are not addressed in some fashion (e.g., semi-implicit or split-explicit temporal differencing). For split-explicit models, the shorter time step used to address sound waves is usually 3-4 times shorter than that used by the rest of the model given a speed of sound approximately 3-4 times faster than the meteorologically-dependent *U*.

Further, vertical advection terms in the primitive equations also pose a constraint on numerical stability, where $U \rightarrow W$ and $\Delta x \rightarrow \Delta z$. Though both *W* and Δz are typically smaller than *U* and Δx, Δz is non-uniform over the model domain, with smaller values near the surface and tropopause and larger values in the middle troposphere. Fortunately, *W* is typically large where Δz is typically large, with the inverse being true as well. However, where *W* is large when Δz is small, such as may be observed with intense vertical circulations within the boundary layer or thunderstorms, the vertical advection term may limit stability more than horizontal advection terms. In practice, it is more often for vertical advection that the CFL criterion is violated in WRF-ARW simulations.

Physical and Mathematical Interpretations of Linear Stability

At its essence, a stability criterion based upon the Courant number indicates that there is a limit to the ratio between the maximum distance that can be traveled in one timestep and the grid spacing. This limit varies as a function of the chosen spatial and temporal differencing schemes used. For the general CFL condition, the maximum distance that can be traveled in one timestep cannot be larger than the grid spacing. For less restrictive differencing schemes, a greater maximum distance relative to the grid spacing can be traveled in one timestep; the opposite is true for more restrictive differencing schemes.

What does this mean, however? Mathematically, the CFL condition can be phrased in terms of the numerical and physical domains of dependence. The former is defined by the finite differencing scheme, while the latter is defined by the underlying meteorology. The numerical domain contains the grid points that contribute to the finite differencing approximated solution, while the physical domain contains the region that contributes to the physical solution. The CFL condition, therefore, states that the chosen differencing scheme will remain numerically stable so long as the numerical domain does not exceed the bounds of the physical domain.

Horizontal Diffusion and Numerical Stability

Another type of forcing term that may exist in the primitive equations is an artificial, or numerical or explicit, diffusion term. Such terms, which dampen or weaken gradients of model variables, can take a form akin to:

$$\frac{\partial h}{\partial t} = K \frac{\partial^2 h}{\partial x^2}$$

For simplicity, the above represents the time tendency due to diffusion only; other terms are said to be implicitly included. Here, K is the diffusion coefficient and the second partial derivative of h is the diffusion operator. K is positive-definite and may be different for horizontal versus vertical diffusion and/or for different model quantities (heat versus momentum). There are other possible diffusion operators, many of them higher-ordered than the above, and we will consider diffusion further in a later lecture.

Recall that the second partial derivative of a field is positive where the field is a local minimum and negative where the field is a local maximum. Thus, for $K > 0$, the diffusion operator above acts to increase low values of h and decrease high values of h; in other words, it weakens the spatial gradient of h. The primary benefit of numerical diffusion lies in its ability to dampen poorly-resolved small-scale features, possibly mitigating aliasing of energy from those wavelengths to the better-resolved larger-scale features (the process of which we will discuss in a later lecture). In some cases, it can also improve numerical stability (e.g., by damping the largest value of U on the model grid).

Diffusion terms, as with advection terms, also influence numerical stability. To illustrate, let us analyze the stability of the second-order diffusion operator using a forward-in-time and second-order-accurate centered-in-space differencing scheme. Though this scheme is always numerically unstable when applied to advection terms, the same is not true for this diffusion operator.

$$\frac{\partial h}{\partial t} = K\frac{\partial^2 h}{\partial x^2} \quad \text{becomes} \qquad \frac{h_x^{t+1} - h_x^t}{\Delta t} = K\left(\frac{h_{x+1}^t + h_{x-1}^t - 2h_x^t}{(\Delta x)^2}\right)$$

If we substitute for h with its wave-like solution given earlier in this lecture, expand the resulting exponential functions, and divide through by a common factor, we obtain:

$$e^{\omega_I \Delta t} e^{-i\omega_R \Delta t} - 1 = \frac{K\Delta t}{(\Delta x)^2}\left(e^{ik\Delta x} + e^{-ik\Delta x} - 2\right)$$

Euler's relations can be added to show that $e^{ik\Delta x} + e^{-ik\Delta x} = 2\cos(k\Delta x)$. Substituting into the above and rewriting the other exponential raised to the power of i using Euler's relations, we obtain:

$$e^{\omega_I \Delta t}\left(\cos(\omega_R \Delta t) - i\sin(\omega_R \Delta t)\right) - 1 = \frac{K\Delta t}{(\Delta x)^2}\left(2\cos(k\Delta x) - 2\right)$$

Splitting this into its real and imaginary components, we obtain:

$$e^{\omega_I \Delta t}\cos(\omega_R \Delta t) - 1 = \frac{K\Delta t}{(\Delta x)^2}\left(2\cos(k\Delta x) - 2\right) \quad \text{(real)}$$

$$-i\sin(\omega_R \Delta t) e^{\omega_I \Delta t} = 0 \ \text{(imaginary)}$$

For a non-zero exponential function in the imaginary equation, the only values of ω_R that satisfy the equality are 0 (such that $\omega_R \Delta t = 0$) and $\Delta t/\pi$ (such that $\omega_R \Delta t = \pi$). It can be shown that this latter case is just a special form of the $\omega_R = 0$ case, and so we focus upon this latter case here. For $\omega_R = 0$, $\cos(\omega_R \Delta t) = 1$ and the real component of the equation becomes:

$$e^{\omega_I \Delta t} = 1 + 2\frac{K\Delta t}{(\Delta x)^2}\left(\cos(k\Delta x) - 1\right)$$

The allowable values of $k\Delta x$ again range from ~0 to π. For $\Delta x \sim 0$, $\cos(k\Delta x) \sim 1$ and $\cos(k\Delta x) - 1 \sim 0$. Thus, $e^{\omega_I \Delta t} = 1$ for all Δt and the solution is numerically-stable. For $k\Delta x = \pi$, representing the $2\Delta x$ wave (given that $\Delta x = L/2$), $\cos(k\Delta x) = -1$ and $\cos(k\Delta x) - 1 = -2$. In this case, the stability criterion equation takes the form:

$$e^{\omega_I \Delta t} = 1 - 4\frac{K\Delta t}{(\Delta x)^2}$$

Because K, Δt, and $(\Delta x)^2$ are all positive-definite, $e^{\omega_I \Delta t} < 1$. But, it is possible for $e^{\omega_I \Delta t} < -1$, which defines exponential growth with a change in the phase of the wave. Note that any negative value defines a change in the phase of the wave; only negative values smaller than -1 denote exponential amplitude growth.

From this, we can assess the stability criterion; simply let $e^{\omega_I \Delta t} = -1$ and change the equality to an inequality (less than or equal to) and rearrange to obtain:

$$-1 \geq 1 - 4\frac{K\Delta t}{(\Delta x)^2} \qquad \text{becomes} \qquad \frac{K\Delta t}{(\Delta x)^2} \leq \frac{1}{2}$$

We can also determine a criterion to prevent the wave from changing phase; simply let $e^{\omega_I \Delta t} = 0$ and rearrange to obtain:

$$0 \geq 1 - 4\frac{K\Delta t}{(\Delta x)^2} \qquad \text{becomes} \qquad \frac{K\Delta t}{(\Delta x)^2} \leq \frac{1}{4}$$

For $\frac{K\Delta t}{(\Delta x)^2} \leq \frac{1}{4}$, the diffusion term is stable with no change in phase. Because Δt and Δx cannot be 0, however, there will always be some damping of the wave's amplitude even if this criterion is met (i.e., from the equation at the top of this page, $e^{\omega_I \Delta t}$ can never exactly be equal to 1).

Returning to the equation at the top of this page, if one were to instead plug in different values for Δx between $L/2$ and 0, the resulting stability criteria would be less stringent than those above. Thus, since Δx cannot be 0, the $2\Delta x$ wave (with $\Delta x = L/2$) is that which limits numerical stability for this diffusion formulation. It can be shown that meeting this criterion produces little damping effect at large wavelengths relative to small wavelengths. This is actually a desirable attribute; the shortest wavelength features are those that are poorly-resolved as it is, and damping them in whole or in part generally helps to keep them from compromising the quality of the simulation.

As with horizontal advection, diffusion also may be formulated in the vertical, with an analogous stability term that must be considered. The total linear stability of the model, then, is limited by the term of the primitive equations that requires in the smallest Δt for a given Δx. This may change throughout the duration of the simulation as the meteorology changes; thus, we typically choose a model time step well below the theoretical limits so as to avoid unnecessarily achieving numerical instability with the chosen model configuration.